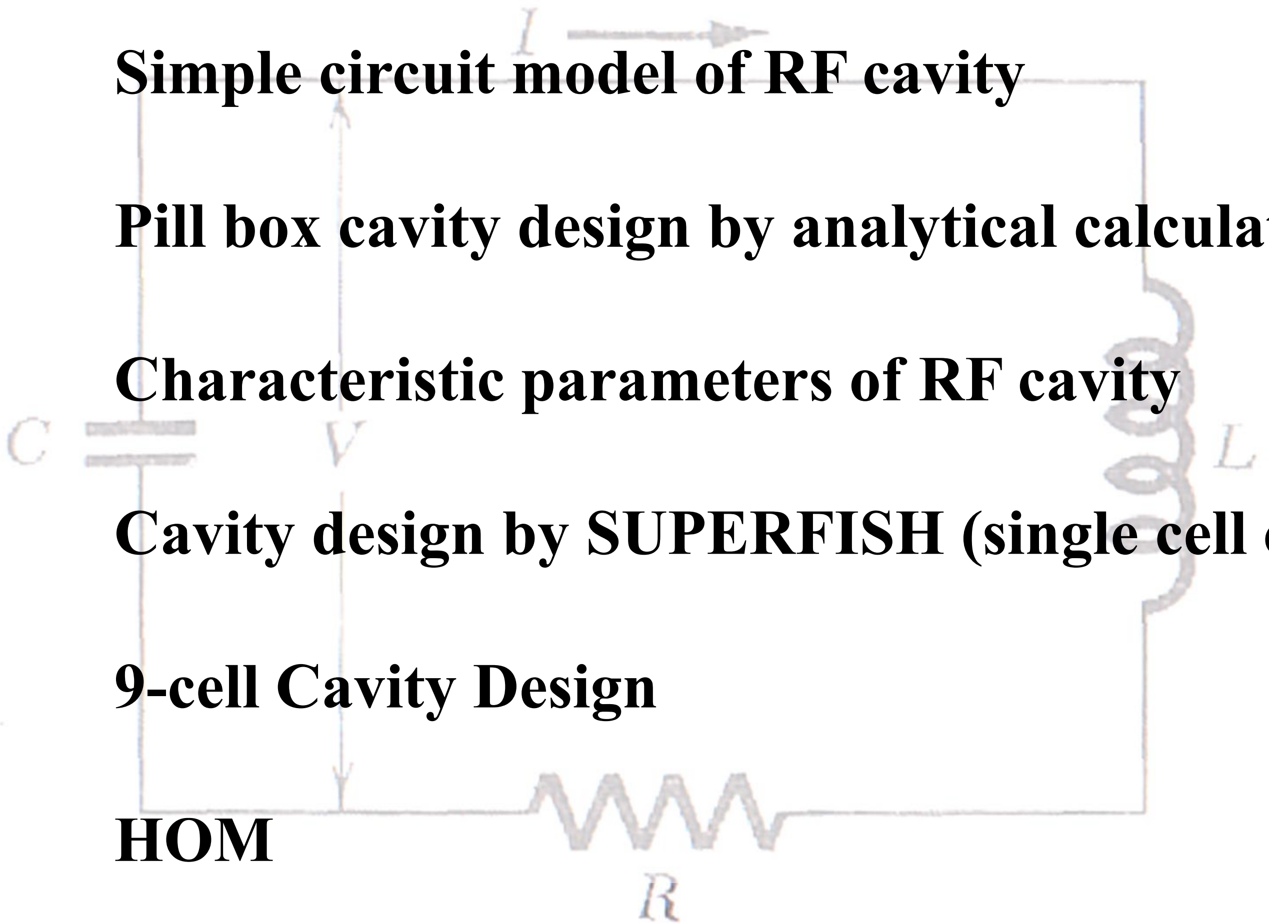
4. SRF 空洞の設計法

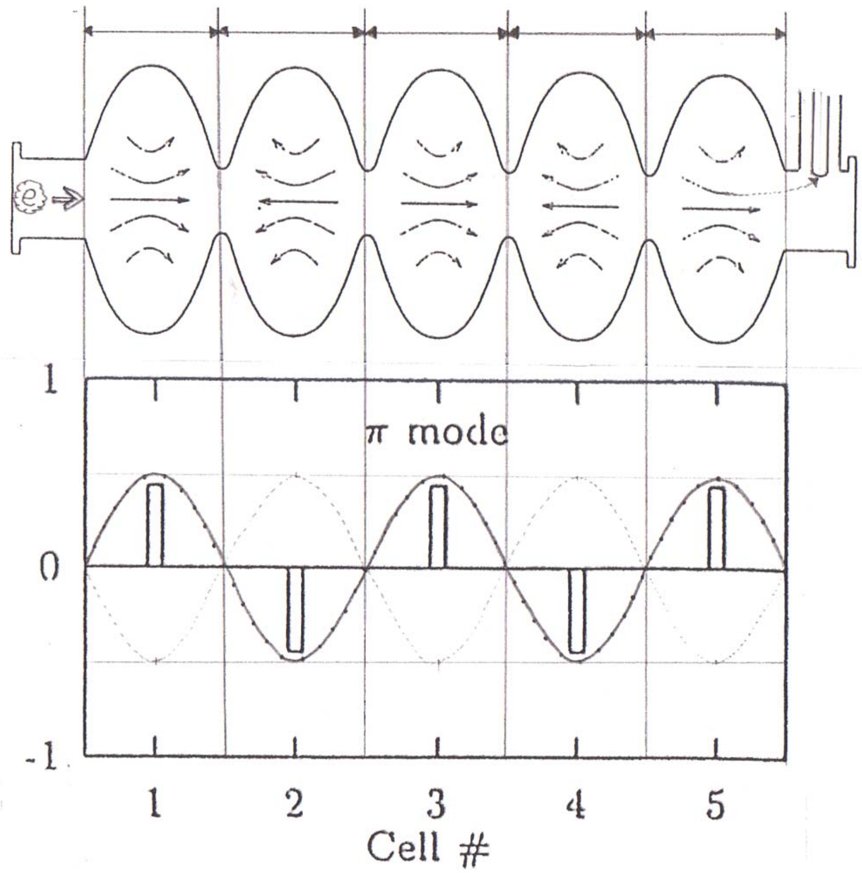
- What is RF cavity
- Simple circuit model of RF cavity
- Pill box cavity design by analytical calculation
- Characteristic parameters of RF cavity
- Cavity design by SUPERFISH (single cell cavity)
- 9-cell Cavity Design
- HOM



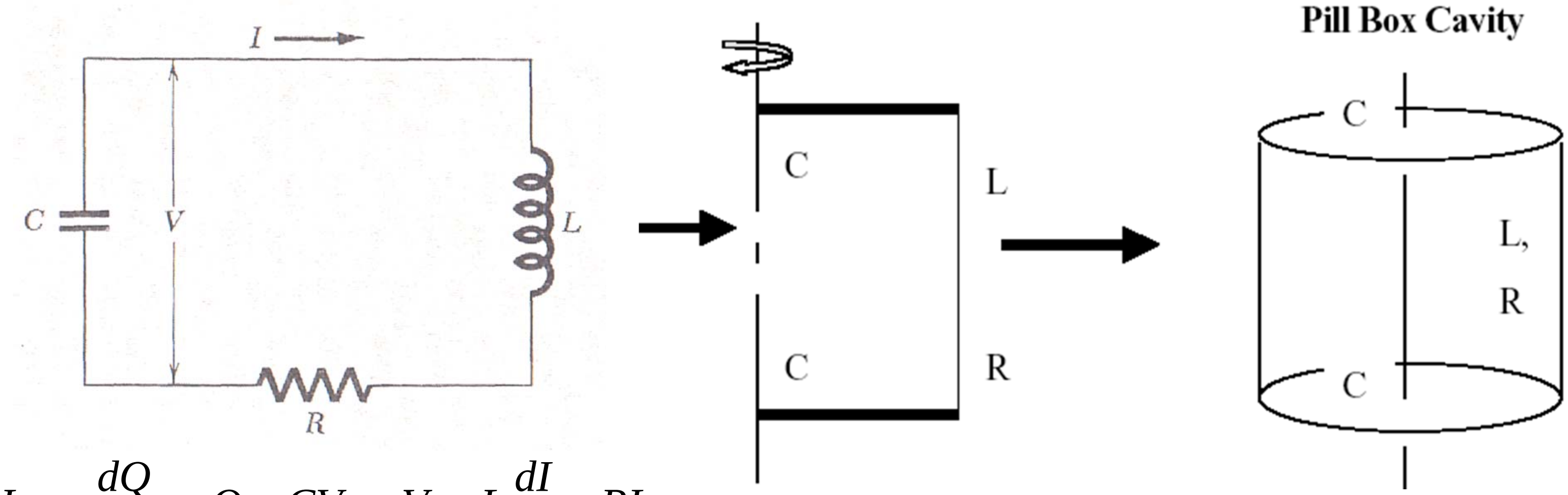
What is RF cavity ?

Principle of RF acceleration
TM-mode : $E_z \neq 0$, $B_z=0$, frequency: f
TM_{010} - mode, π-mode, Standing Wave
V(electron velocity)～C(light velocity)
L(cell length) = $\lambda/2$; λ(wave length)=C/f
If the velocity is low like protons,
$\beta=V/C < 1$, then $L=\beta\lambda/2$

RF Cavity: accelerates charged particles by the electric field synchronized with RF frequency.



Equivalent circuit



$$I = -\frac{dQ}{dt}, \quad Q = CV, \quad V = L\frac{dI}{dt} + RI$$

$$\frac{d^2V}{dt} + \left(\frac{R}{L}\right)\frac{dV}{dt} + \left(\frac{V}{LC}\right) = 0, \quad V(t) = V_O \exp(-\alpha + i\omega)t$$

$$(-\alpha + i\omega)^2 + (-\alpha + i\omega)\left(\frac{R}{L}\right) + \left(\frac{1}{LC}\right) = 0,$$

$$\alpha = \frac{R}{2L}, \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$R << L, \quad \omega_O^2 = \frac{1}{LC} \Rightarrow f = \frac{1}{2\pi\sqrt{LC}}$$

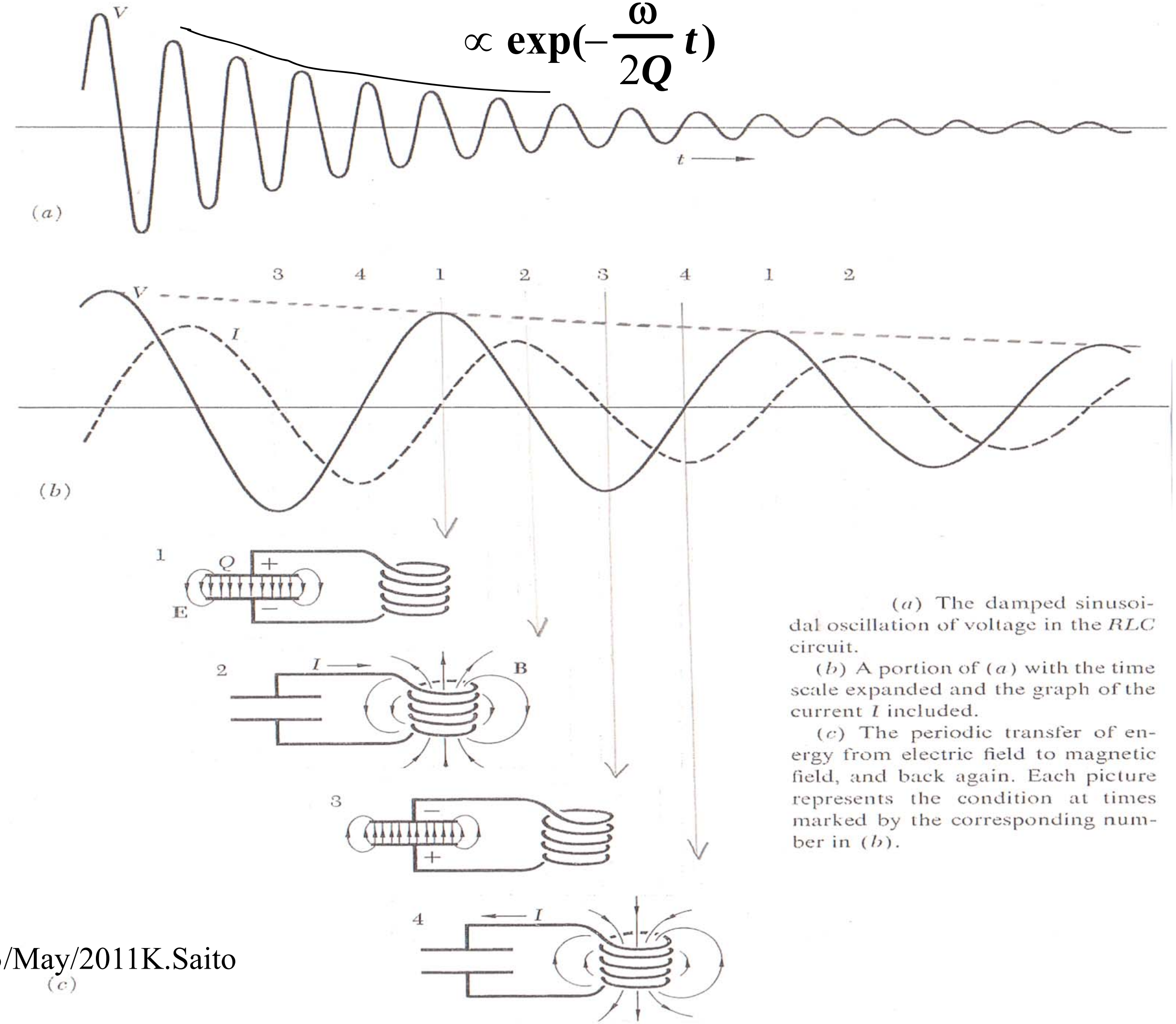
Q-value of the circut

$$\boldsymbol{Q} \equiv \omega \cdot \frac{\textbf{\textit{stored energy}}}{\textbf{\textit{power loss}} \,/\, \textbf{sec}} = \omega \cdot \frac{\boldsymbol{P}}{\boldsymbol{dP/dt}} = \omega \cdot \frac{\boldsymbol{L}}{\boldsymbol{R}}$$

$$= \frac{\omega}{2\alpha}$$

Simple Circuit Model of RF Cavity
- Oscillation in the LCR Circuit -

$$\propto \exp\left(-\frac{\omega}{2Q}t\right)$$



(*a*) The damped sinusoidal oscillation of voltage in the *RLC* circuit.
(*b*) A portion of (*a*) with the time scale expanded and the graph of the current *I* included.
(*c*) The periodic transfer of energy from electric field to magnetic field, and back again. Each picture represents the condition at times marked by the corresponding number in (*b*).

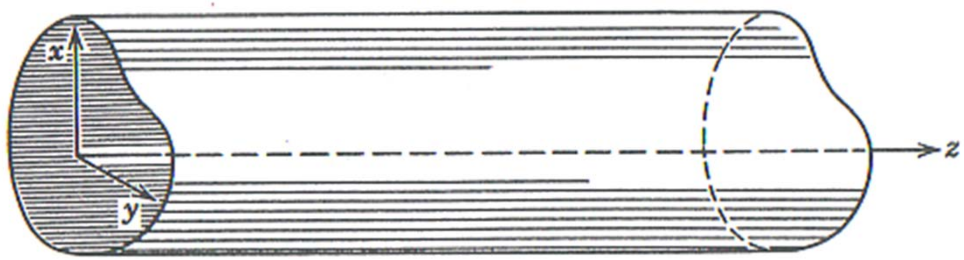
4.1 Maxwell方程式を使って解析的に扱える Pill Box空洞を設計する

Electro-magnetic field in a waveguide

Maxwell equations in a waveguide

$$\vec{\nabla}\times\vec{E} = i\frac{\omega}{c}\vec{B},\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{B}=-i\mu\varepsilon\frac{\omega}{c}\vec{E},\quad \vec{\nabla}\cdot\vec{E}=0,\quad \rho=0,\ j=0$$

$$\left(\vec{\nabla}^2+\mu\varepsilon\frac{\omega^2}{c^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix}=0,$$



$$\vec{E}(x,y,z,t)=\vec{E}(x,y)\exp(\pm ikz-i\omega t),\quad k\text{: wave vector}$$

$$\vec{B}(x,y,z,t)=\vec{B}(x,y)\exp(\pm ikz-i\omega t)$$

$$\left[\vec{\nabla}_t^2+\left(\varepsilon\mu\frac{\omega^2}{c^2}-k^2\right)\right]\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix}=0,\quad \vec{\nabla}_t^2\equiv\vec{\nabla}^2-\frac{\partial^2}{\partial^2 z},\quad \vec{E}=E_z\vec{e}_z+\vec{E}_t,\quad \vec{B}=B_z\vec{e}_z+\vec{B}_t$$

$$\vec{B}_t=\frac{1}{\left(\mu\varepsilon\frac{\omega^2}{c^2}-k^2\right)}\left[\vec{\nabla}_t\left(\frac{\partial B_z}{\partial z}\right)+i\mu\varepsilon\frac{\omega}{c}\vec{e}_z\times\vec{\nabla}_t E_z\right],$$

$$\vec{E}_t=\frac{1}{\left(\mu\varepsilon\frac{\omega^2}{c^2}-k^2\right)}\left[\vec{\nabla}_t\left(\frac{\partial E_z}{\partial z}\right)-i\frac{\omega}{c}\vec{e}_z\times\vec{\nabla}_t B_z\right]$$

TM/TE - mode assignment of electro-magnetic field

TM-mode : $\boldsymbol{B_z} = 0,\ \boldsymbol{E_z} \neq 0$ ⇒ Beamを加速出来る。

$$\vec{B}_t = \frac{i\varepsilon\mu\frac{\omega}{c}}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)}\vec{e}_z \times \vec{\nabla}_t E_z$$

$$\vec{E}_t = \frac{1}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)}\vec{\nabla}_t\left(\frac{\partial E_z}{\partial z}\right)$$

⇒ 固有値問題を解いてk, Ezを求める。

$$\vec{\nabla}_t^2 E_z + \left(\varepsilon\mu\frac{\omega^2}{c^2} - k^2\right)E_z = 0\ ,$$

Boundary condition $E_z|_S = 0$ **(**$\because\ \vec{n}\times\vec{E} = 0$ **on the sutface of perfect conductor)**

$\frac{\partial B_z}{\partial n}|_S = 0$ **(**$n \cdot B = 0$ **on the surface,**

but automatically satisfied by the TM - mode condition)

TE-mode : $E_z = 0$, $B_z \neq 0$ ⇒ Beamをキックする。

$$\vec{B}_t = \frac{i\varepsilon\mu \frac{\omega}{c}}{(\varepsilon\mu \frac{\omega^2}{c^2} - k^2)} \vec{\nabla}_t \left(\frac{\partial B_z}{\partial z} \right)$$

$$\vec{E}_t = \frac{-i \frac{\omega}{c}}{(\varepsilon\mu \frac{\omega^2}{c^2} - k^2)} \vec{e}_z \times \vec{\nabla}_t B_z$$

$$\vec{\nabla}_t^2 B_z + \left(\varepsilon\mu \frac{\omega^2}{c^2} - k^2 \right) B_z = 0 \ ,$$

Boundary condition $E_z|_S = 0$ ($\vec{n} \times \vec{E} = 0$ **on the surface of perfect conductor but automatically satisfied by the TE- mode condition)**

$$\frac{\partial B_z}{\partial n}\Big|_S = 0 \ (\because n \cdot B = 0 \ \text{on the surface})$$

固有値問題

$\psi(x,y) = E_z(x,y)$ for TM mode or $B_z(x,y)$ for TE mode

$$\left(\vec{\nabla}_t^2 + \gamma^2\right)\psi = 0, \quad \psi|_S = 0 \text{ (for TM - mode) or } \frac{\partial}{\partial n}\psi|_S = 0 \text{ (for TE - mode)}$$

$$\gamma^2 = \varepsilon\mu\frac{\omega^2}{c^2} - k^2 \geq 0$$

From the boundary condition,

$$\gamma^2 = \gamma_\lambda^2, \ \psi = \psi_\lambda \quad (\lambda = 1, 2, \cdots)$$

$$\Rightarrow \quad k_\lambda^2 = \varepsilon\mu\frac{\omega^2}{c^2} - \gamma_\lambda^2$$

If $\omega < c\frac{\gamma_\lambda}{\sqrt{\varepsilon\mu}}$, then k_λ is an imaginal number. The wave is damped in the waveguic

$$\omega_\lambda = c\frac{\gamma_\lambda}{\sqrt{\varepsilon\mu}} \cdots \text{cutoff frequency}$$

When $\omega \geq \omega_\lambda$, wave number k_λ is a real number,

then the wave can propagate into the waveguide.

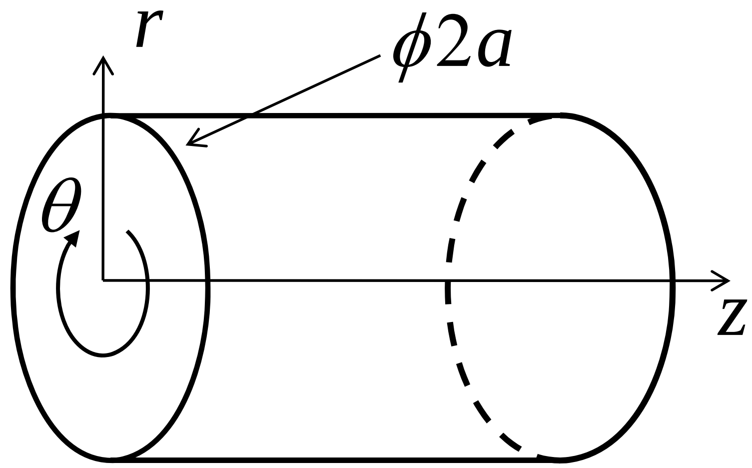
具体的な応用として円筒導波管の計算

円筒導波管を進む電磁場について考える

z方向に進む平面波、位相の進む速度を表す

$$\begin{cases} \vec{E} = \vec{E}_0 e^{-i(kz-\omega t)} \\ \vec{H} = \vec{H}_0 e^{-i(kz-\omega t)} \end{cases}$$

k:虚数⇒波の減衰
k:実数⇒波の伝搬



ω: 角周波数　k: 波数

進行方向(longitudinal)と垂直方向(transverse) に分け,
Maxwell方程式に代入

$$\Delta \vec{E}_z + \gamma^2 \vec{E}_z = 0 \qquad \Delta \vec{H}_z + \gamma^2 \vec{H}_z = 0 \qquad \gamma^2 = \omega^2 \varepsilon_0 \mu_0 - k^2$$

(γ: 伝搬定数)

γの固有値問題　⇒γを決めると $k^2 = \omega^2 \varepsilon_0 \mu_0 - \gamma^2$ より波が伝搬するためのωの条件が求まる

$$\vec{E}_j = -\frac{1}{\gamma^2}\left(-i\omega\mu_0 \cdot \vec{z} \times \vec{\nabla} H_z + k \cdot \vec{\nabla} E_z\right)_j \qquad \vec{H}_j = -\frac{1}{\gamma^2}\left(k \cdot \vec{\nabla} H_z + i\omega\varepsilon_0 \cdot \vec{z} \times \vec{\nabla} E_z\right)_j$$

jはxまたはy

①　Et,Htともに、Ez,Hzのみで決まる
②　Hz=0,Ez≠0 の場合　EtとHtはEzのみで書ける　（TMモード，加速モード）
③　Hz≠0,Ez=0 の場合　EtとHtはHzのみで書ける　（TEモード，キックモード）

TM Mode

円筒導波管がTM Modeで伝搬するときの条件を導く。

円柱座標で表す。

$$\nabla_t^2 E_z + \gamma^2 E_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) E_z = 0$$

$E_z = R(r)\Theta(\theta)$ と仮定する ➡ $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) R(r)\Theta(\theta) = 0$

➡ $\frac{r^2}{R(r)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \gamma^2 \right) R(r) = -\frac{1}{\Theta(\theta)}\frac{\partial^2}{\partial \theta^2}\Theta(\theta) \quad \left(= \nu^2 : 定数 \right)$

ここで、ρ=γrとおくと**Bessel**の**微分方程式**になる。

Bessel微分方程式: $\frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left(1 - \frac{\nu^2}{\rho^2} \right) R(\rho) = 0$

$\frac{d^2\Theta(\theta)}{d\theta^2} + \nu^2\Theta(\theta) = 0$ **単振動** 一周期で連続一価関数⇒mは整数

一般解は

$$R(r) \propto J_m(\gamma r), \quad \Theta(\theta) \propto e^{im\theta}$$

最初の式は $E_z = E_0 e^{im\theta} J_m(\gamma r) e^{-i(kz - \omega t)}$ とおける。

境界条件(円筒導波管の場合)

完全導体表面、
$E_\perp \neq 0$,$E_{//}=0$, $H_\perp=0$, $H_{//}\neq 0$
⇒γが決まる

導波管表面に平行な電場がゼロでなければならない。

$r=a$のときに $J_m(\gamma a)=0$

$$\Rightarrow \gamma a=\rho_{mn} \Rightarrow \gamma=\frac{\rho_{mn}}{a}$$

$$\Rightarrow \quad \gamma r = \frac{\rho_{mn}}{a} r$$

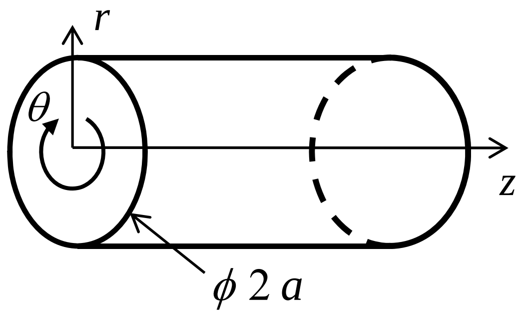
円筒導波管中のEzの解

$$E_z = E_0 e^{im\theta} J_m\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$

m次のBessel関数の解

ρ_{mn}はn個目の根

m: Θ方向の節の数
n: r方向の節の数

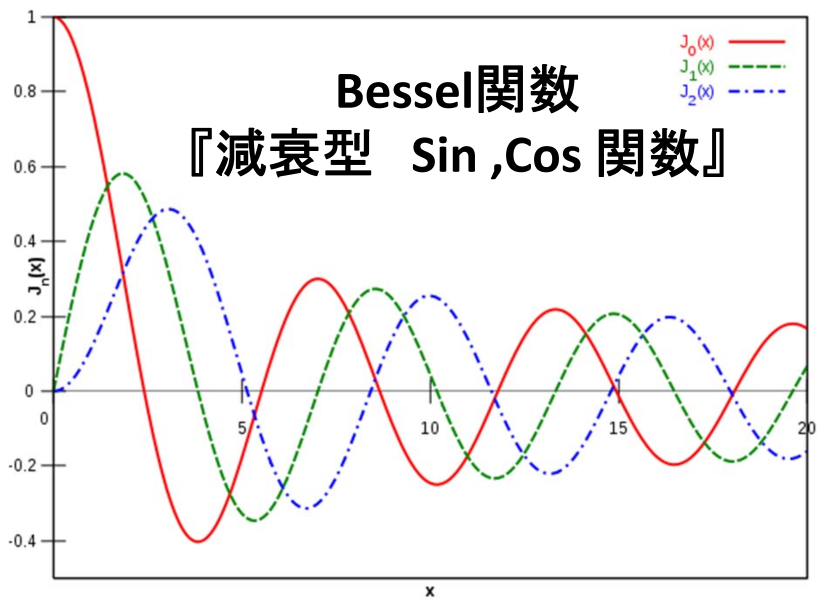


ErとEθを求めるにはEzをMaxwell方程式に代入すればよい

$$\vec{E}_t = -\frac{1}{\gamma^2}\left(-i\omega\mu_0 \cdot \nabla_t \times H_z + k \cdot \nabla_t E_z\right)$$

$$E_r = \frac{ik}{\gamma^2}\frac{\partial E_z}{\partial r} = -\frac{ik}{\gamma^2}\frac{\rho_{mn}}{a} E_0 e^{im\theta} J_m{}'\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$

$$E_\theta = \frac{ik}{\gamma^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta} = \frac{ik}{\gamma^2}\frac{im}{r} E_0 e^{im\theta} J_m\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$



円筒導波管　TM Mode の　Cutoff周波数

- 波数kが実数でないと波：$\vec{E} = \vec{E}_0 e^{-i(kz-\omega t)}$ は伝搬できない
- 波動方程式の固有値γは境界条件から　$\gamma = \dfrac{\rho_{mn}}{a}$

⇒ 波が伝搬するためのωの条件を求める。

$$\gamma^2 = \omega^2 \varepsilon_0 \mu_0 - k^2 \Rightarrow k^2 = \omega^2 \varepsilon_0 \mu_0 - \gamma^2 > 0 \Rightarrow \omega^2 > \frac{1}{\varepsilon_0 \mu_0}\gamma^2 = c^2 \left(\frac{\rho_{mn}}{a}\right)^2$$

⇒ $\omega = \dfrac{c\rho_{mn}}{a}$　この限界の周波数をCutoff周波数と呼ぶ

Cutoff周波数： $f_{Cutoff} = \dfrac{c\rho_{mn}}{2\pi a}$

φ60mm円筒導波管のTM_{mn}ModeのCutoffを計算する。

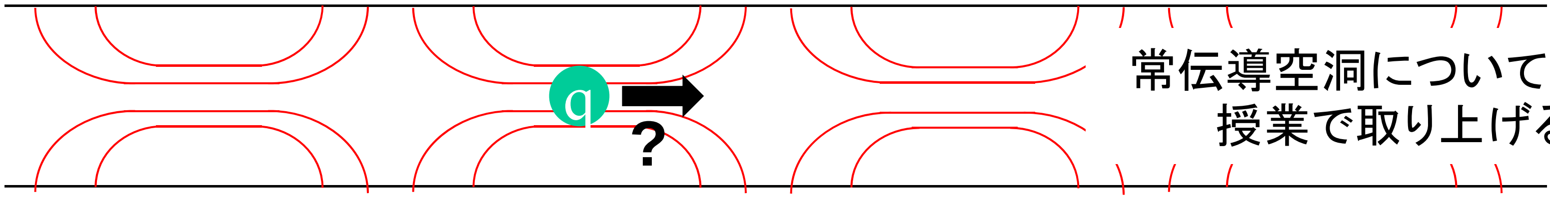
Mode	Cutoff Freqency[MHz]
TM01	3825
TM11	6095
TM21	8169
TM22	8779
TM21	11159
TM22	13387

$J_m(x)$のn番目のゼロ点 $x=\rho_{mn}$

m \ n	1	2	3
0	2.405	5.520	8.654
1	3.832	7.016	10.173
2	5.136	8.417	11.620

円筒導波管で加速できるか？

導波管にTM01 Modeを励起した場合にビーム加速は可能か？



常伝導空洞については後期授業で取り上げる。

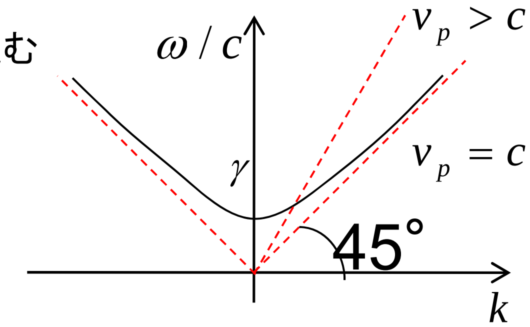
TM01は $E_z = E_0 J_0\left(\frac{\rho_{01}}{a} r\right) e^{-i(kz-\omega t)}$ で波が進む(波が伝搬するためにはkが実数)

kとωは導波管形状で決まるγとの関係がある $\gamma^2 = \omega^2\varepsilon_0\mu_0 - k^2 = \left(\frac{\omega}{c}\right)^2 - k^2$ ➡ $\begin{cases} k^2 = \left(\frac{\omega}{c}\right)^2 - \gamma^2 \\ \left(\frac{\omega}{c}\right)^2 = k^2 + \gamma^2 \end{cases}$

波が伝搬するとき、γはω/cよりも小さい

$kz = \omega t$ のとき最大電場が発生

位相速度: $v_p = \frac{z}{t} = \frac{\omega}{k} = \frac{\omega}{\sqrt{(\omega/c)^2 - \gamma^2}} > c$ で最大電場位置が進む



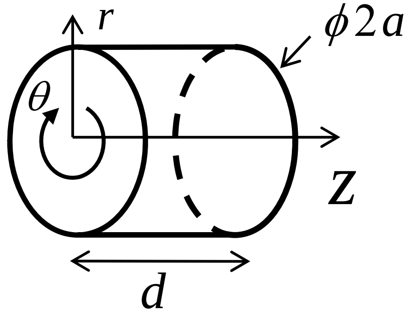
つまり光速で走る粒子よりも電磁場の位相の進みが早い

➡ 加速できない or 効率が悪い

境界条件(PillBoxの場合)

円筒形導波管の両端に蓋をする(Cavity) ➡ 両端板(Z方向)での境界条件が追加される

端板の位置： $z=0,d$ で $E_t=0 \quad E_r=E_\theta=0$



端板に水平な電場成分として例えばR成分は

$$E_r=-\frac{ik}{\gamma^2}\frac{\rho_{mn}}{a}E_0e^{im\theta}J_m'\left(\frac{\rho_{mn}}{a}r\right)e^{-ikz}=-A\sin kz+iA\cos kz$$

実数成分のみを考え ➡ $\mathrm{Re}[E_r(z)]=-A\sin kz$

$$\mathrm{Re}[E_r(0)]=-A\sin 0=0$$
$$\mathrm{Re}[E_r(d)]=-A\sin kd$$

$\rightarrow \sin kd=0\Rightarrow kd=p\pi\Rightarrow k=\dfrac{p\pi}{d}$ (pは整数)

つまりZ方向は定在波を表し、 pはz方向の節の数になっている

TM$_{mnp}$モード 実数部のみを考えればよい

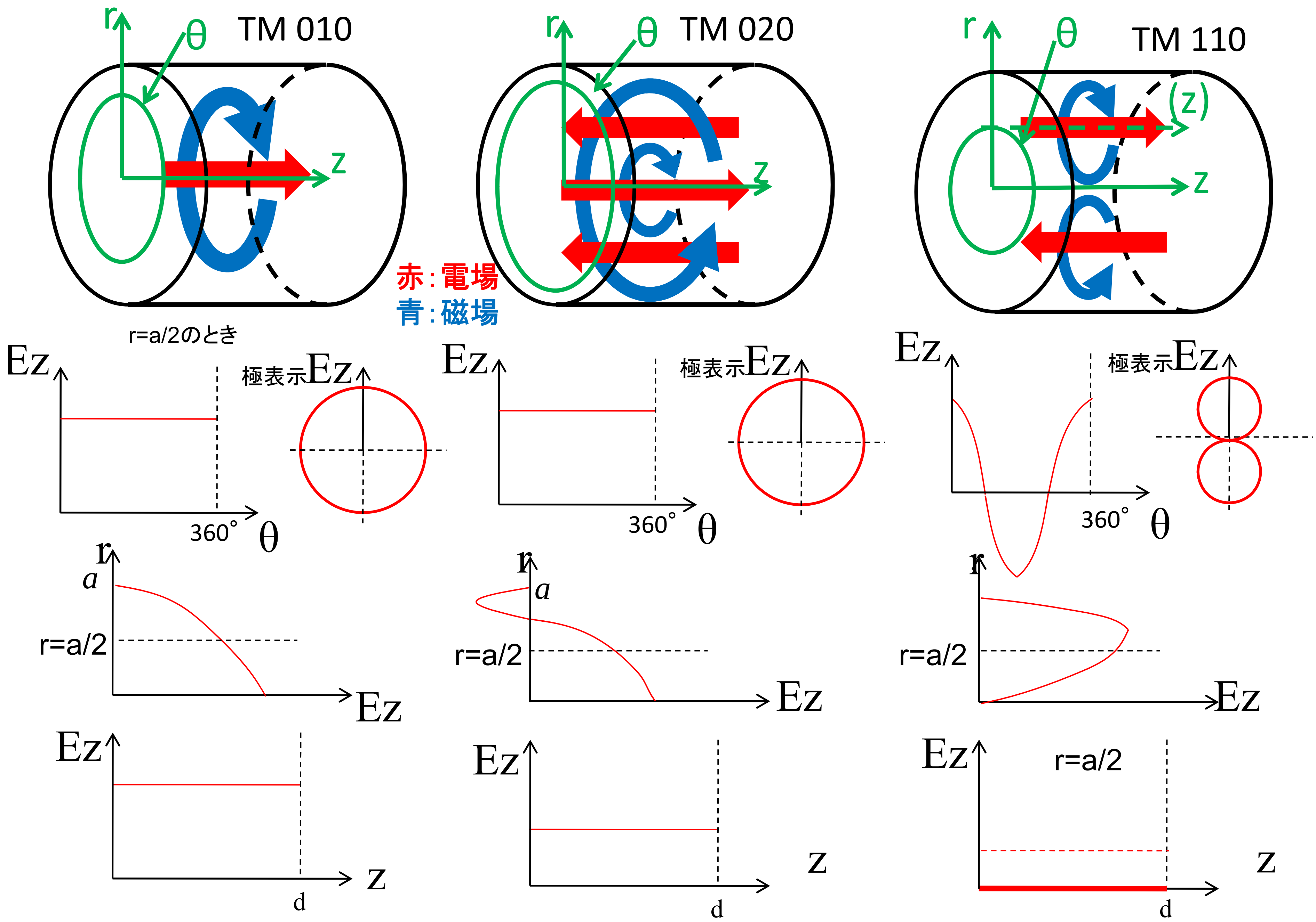
$$E_r=-\frac{1}{\gamma^2}\frac{p\pi}{d}\frac{\rho_{mn}}{a}E_0J_m'\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\sin(\frac{p\pi z}{d}) \qquad H_r=-\frac{\omega\varepsilon_0}{\gamma^2}\frac{m}{r}E_0J_m\left(\frac{\rho_{mn}}{a}r\right)\sin(m\theta)\cos(\frac{p\pi z}{d})$$

$$E_\theta=\frac{1}{\gamma^2}\frac{p\pi}{d}\frac{m}{r}E_0J_m\left(\frac{\rho_{mn}}{a}r\right)\sin(m\theta)\sin(\frac{p\pi z}{d}) \qquad H_\theta=-\frac{\omega\varepsilon_0}{\gamma^2}\frac{j_{mn}}{a}E_0J_m'\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\cos(\frac{p\pi z}{d})$$

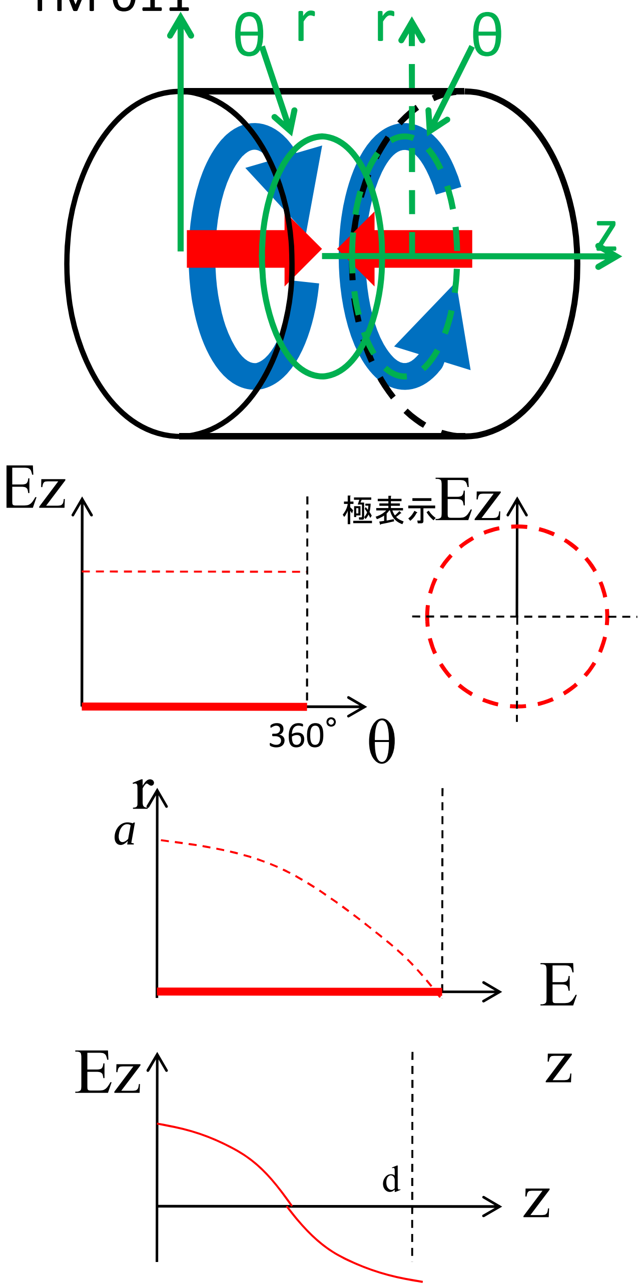
$$E_z=E_0J_m\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\cos(\frac{p\pi z}{d}) \qquad H_z=0$$

共振周波数 $\omega^2\varepsilon\mu=k^2+\gamma^2=\left(\dfrac{\rho_{mn}}{a}\right)^2+\left(\dfrac{p\pi}{d}\right)^2$

TM_{mnp}モードの例　その1



TM$_{mnp}$モードの例　その2



課題①（宿題）

Pillboxで以下のTM Modeの分布を考えよう。

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos(\frac{p\pi z}{d})$$

Θ方向 : $\cos(m\theta)$　R方向 : $J_m\left(\frac{\rho_{mn}}{a} r\right)$　Z方向 : $\cos(\frac{p\pi z}{d})$

TM010
TM011
TM110
TM111
TM210
TM012
TM020

Beampipe

TM Mode　周波数

$$\omega^2\varepsilon_0\mu_0 = \left(\frac{\rho_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \Rightarrow f = \frac{c}{2\pi}\sqrt{\left(\frac{\rho_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

d は任意だが、多連空洞で加速する場合
電子が光速で走っているとして設計する
d の長さを半波長に取ることで常に加速位相に
電子を乗せることができる。π-Mode(斎藤講義)

課題②

TM010モードの共振周波数が1300MHzになるためのPillboxの半径aと長さdを求めよう。

$$a = \frac{c\rho_{01}}{2\pi f} = \quad [mm] ,\quad d = \frac{c}{2f} = \quad [mm]$$

$$c = 2.99792458 \times 10^8 [m/s]$$

$J_m(x)$のn番目のゼロ点　$x=\rho_{mn}$

m \ n	1	2	3
0	2.405	5.520	8.654
1	3.832	7.016	10.173
2	5.136	8.417	11.620

課題③

課題②で得たPillbox形状のとき、他のTM Modeの共振周波数を計算してみよう。
また、半径30mmのBeampipeのCutoff周波数と比べて、FieldがBeampipeに漏れ出すか考えよう

TM mnp	f [MHz]	Beampipe
TM 0 1 0		
TM 0 1 1		
TM 1 1 0		
TM 1 1 1		
TM 2 1 0		
TM 0 1 2		
TM 0 2 0	2984	Reject

TM mnp	f [MHz]	Beampipe
TM 0 1 3		
TM 1 1 5		
TM 2 1 6		

低周波数ModeはCavity内に閉じ込められる
これをTrapped Modeという。

半径30mmのBeampipeのCutoff

Mode	Cutoff [MHz]
TM01	
TM11	
TM21	8169

TE Mode

Ez=0の場合をTEモードという、TM Modeと同様にしてHzに着目しつつ解いていく

円筒導波管でまず考える $\nabla_t^2 H_z + \gamma^2 H_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) H_z = 0$

R成分とθ成分を変数分離して解いてみると、TM Modeと同様に平面波は次式のように表せる。

$$H_z = H_0 e^{im\theta} J_m(\gamma r) e^{-i(kz-\omega t)}$$

ただし、境界条件は 円筒導体表面に垂直な磁場がゼロになること なので

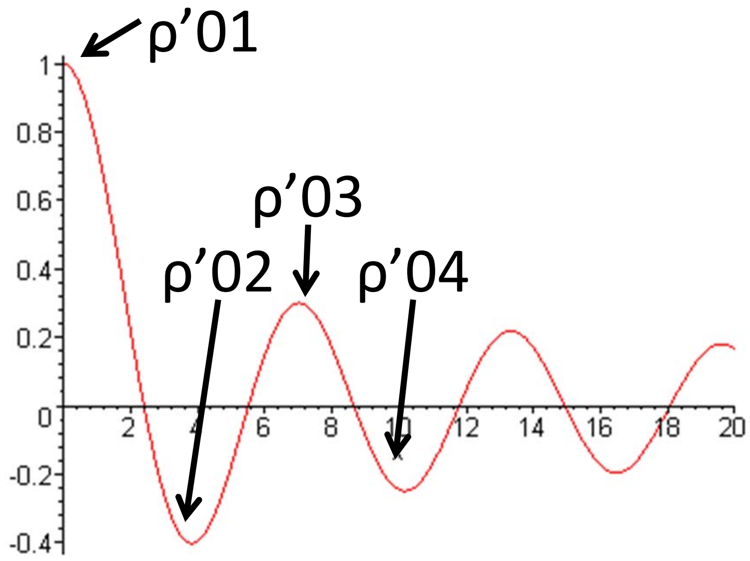
$$H_r\Big|_{r=a} = \frac{ik}{\gamma^2}\frac{\partial H_z}{\partial r} = \frac{ik}{\gamma^2} H_0 k e^{im\theta} J_m'(\gamma r)\Big|_{r=a} = 0$$

$J_m'(x)$ ($J_m(x)$の微分)のn番目のゼロ点をx=ρ_{mn}'とすると

$$H_z = H_0 J_m\left(\frac{\rho'_{mn}}{a} r\right) e^{im\theta} e^{-i(kz-\omega t)}$$

なお、境界条件よりTE Modeの伝搬定数γは

$$\gamma = \frac{\rho'_{mn}}{a}$$



円筒導波管 TE Mode の Cutoff周波数

位相定数kが実数でないと波 $:\vec{E}=\vec{E}_0 e^{-i(kz-\omega t)}$ は伝搬できない

波動方程式の固有値γは $\gamma = \dfrac{\rho'_{mn}}{a}$

波が伝搬するためのωの条件を求める。

$$\gamma^2=\omega^2\varepsilon_0\mu_0-k^2 \Rightarrow k^2=\omega^2\varepsilon_0\mu_0-\gamma^2>0 \Rightarrow \omega^2>\frac{1}{\varepsilon_0\mu_0}\gamma^2=c^2\left(\frac{\rho'_{mn}}{a}\right)^2$$

$\Rightarrow \omega > \dfrac{c\rho'_{mn}}{a}$ この限界の周波数をCutoff周波数と呼ぶ

Cutoff周波数 : $f_{Cutoff}=\dfrac{c\rho'_{mn}}{2\pi a}$

φ60mm円筒導波管のTE$_{mn}$ModeのCutoffを計算する

Mode	Cutoff Freqency[MHz]
TE11	2928
TE21	4857
TE01	6094
TE12	8478
TE22	10665
TE02	11158

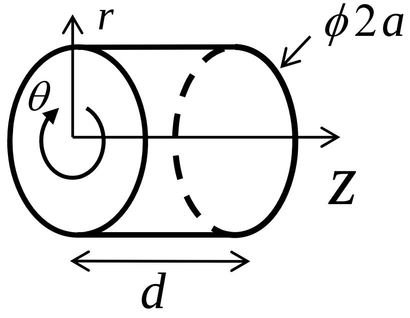
J’$_m$(x)のn番目のゼロ点 x=ρ’$_{mn}$

m \ n	1	2	3
0	3.832	7.016	10.173
1	1.841	5.331	8.536
2	3.054	6.706	9.969

PillBox 内のTE Mode電磁場分布

TM Modeと異なり、TE Modeの境界条件は導体表面で垂直な磁場がゼロになる

➡ 端板の位置: $z=0,d$ で $H_z=0$



$$H_z = H_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)e^{-ikz} = -A\sin kz + iA\cos kz$$

実数成分のみを考え ➡ $\mathrm{Re}[E_r(z)] = -A\sin kz$

$$\mathrm{Re}[H_z(0)] = -A\sin 0 = 0$$

$$\mathrm{Re}[E_z(d)] = -A\sin kd$$

$\rightarrow \sin kd = 0 \Rightarrow kd = p\pi \Rightarrow k = \dfrac{p\pi}{d}$ (pは整数)

つまりZ方向は定在波を表し、 pはz方向の腹の数になっている
p=0は存在しない、なぜならE=0になるため

$\mathrm{TE_{mnp}}$モード 実数部のみを考えればよい

$$E_r = \frac{\omega\varepsilon}{\gamma^2}\frac{m}{r}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\sin(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

$$E_\theta = \frac{\omega\varepsilon}{\gamma^2}\frac{\rho'_{mn}}{a}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

$$E_z = 0$$

$$H_r = \frac{1}{\gamma^2}\frac{p\pi}{d}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

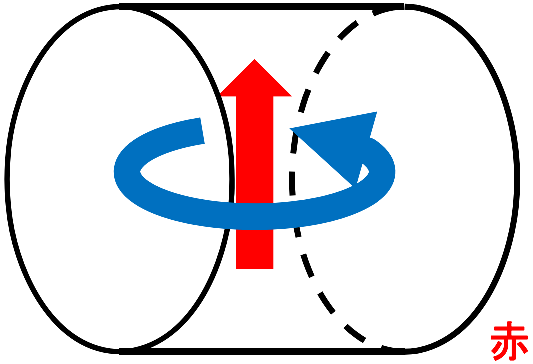
$$H_\theta = -\frac{1}{\gamma^2}\frac{p\pi}{d}\frac{m}{r}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\sin(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

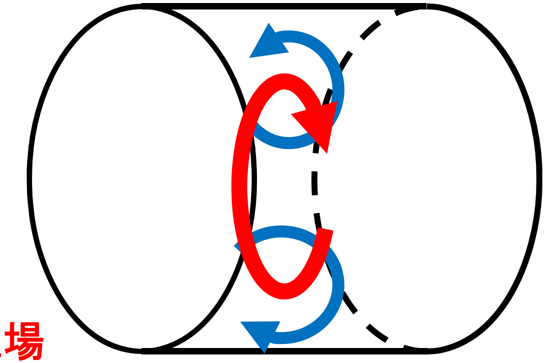
共振周波数 $\omega^2\varepsilon_0\mu_0 = \left(\dfrac{\rho'_{mn}}{a}\right)^2 + \left(\dfrac{p\pi}{d}\right)^2 \Rightarrow f = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{\rho'_{mn}}{a}\right)^2 + \left(\dfrac{p\pi}{d}\right)^2}$

TE_{mnp}モードの例

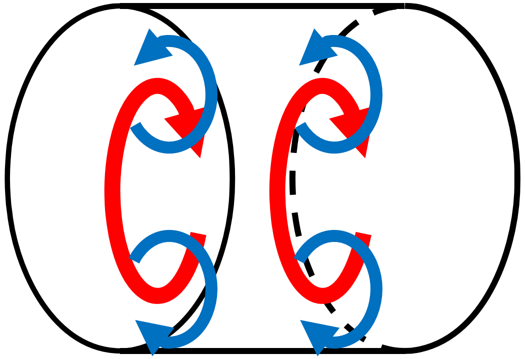
TE 111



TE 011



TE 012



赤:電場
青:磁場

課題④(宿題) PillboxのTE111、TE211、TE011、TE112の　電磁場分布を考えてみよう

課題⑤(宿題)

課題②で得たPillbox形状のとき、TEModeの共振周波数を計算してみよう。
また、半径30mmのBeampipeのCutoff周波数と比べて、FieldがBeampipeに漏れ出すか考えよう

$$f = \frac{c}{2\pi}\sqrt{\left(\frac{\rho'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$J'_m(x)$のn番目のゼロ点 $x=\rho'_{mn}$

m \ n	1	2	3
0	3.832	7.016	10.173
1	1.841	5.331	8.536
2	3.054	6.706	9.969

TE mnp	f [MHz]	Beampipe
TE 1 1 1		
TE 2 1 1		
TE 0 1 1		
TE 1 1 2		
TE 1 1 3		
TE 2 1 4		
TE 0 1 5		

半径30mmのBeampipeのCutoff

Mode	Cutoff [MHz]
TE11	2928
TE21	4857
TE01	6094

Modeの同定方法　その1

TM_{mnp} mode :Hz=0,Ez≠0 の場合

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

m：θ方向の節の数
n：　r方向の節の数(中心は含まない)
p：　z方向の節の数

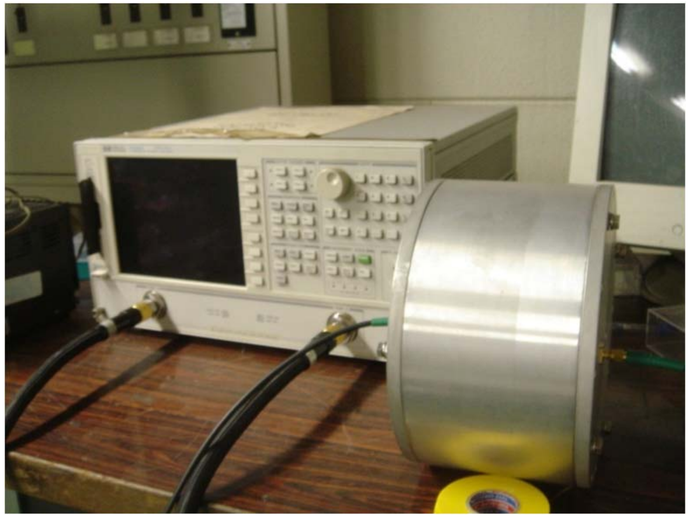
TE_{mnp} mode:Hz≠0,Ez=0　の場合

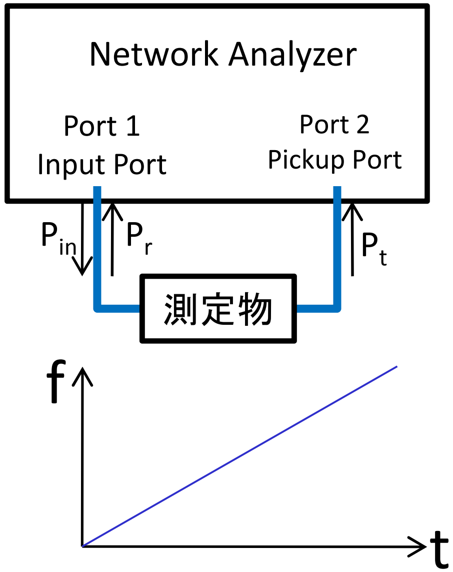
$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a} r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

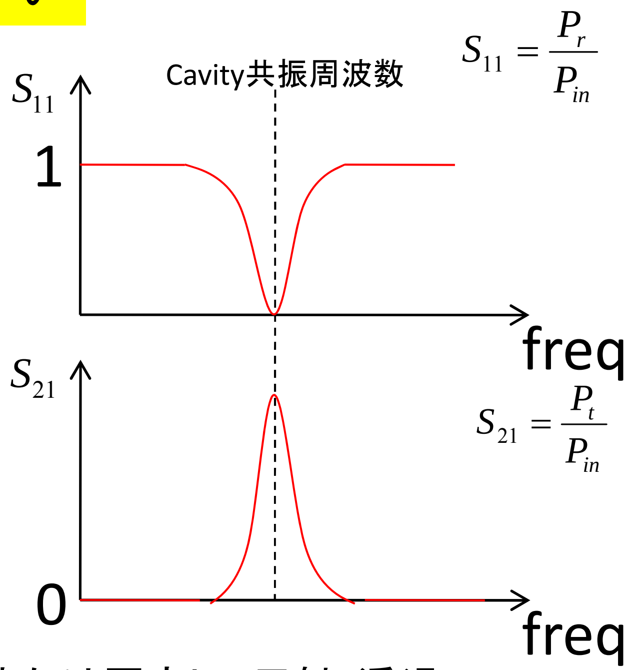
m：θ方向の腹の数
n：　r方向の腹の数
p：　z方向の腹の数

それぞれの軸に沿っての分布が分かればいい

Network Analyzer



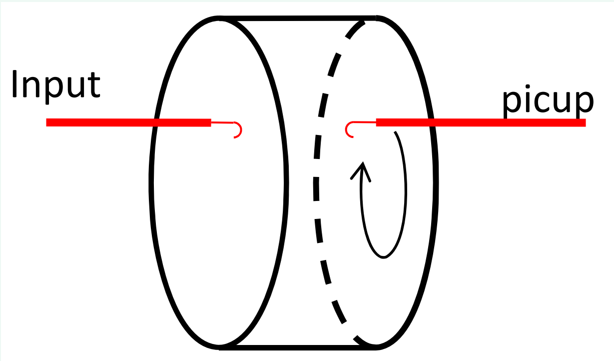


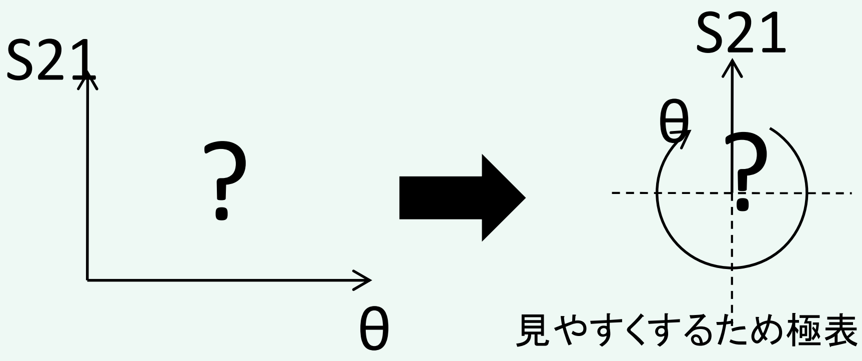


Network Analyzerは入力パワーを一定にして、周波数を掃引または固定し、反射・透過のパワー・位相を様々な形式で読み取ってくれる測定器である。
今回はSパラメータと位相表示を使う。空洞にアンテナを挿入し、NAからRF場を励起する。

Modeの同定方法　その2

Θ方向の分布を見る:　Input antennaを固定し、
Pick upアンテナをθ方向に回しS21をモニタする



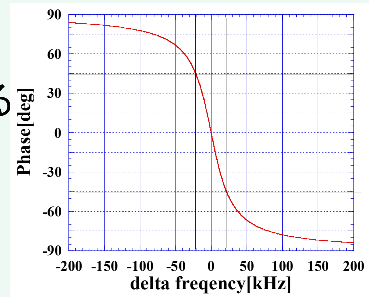


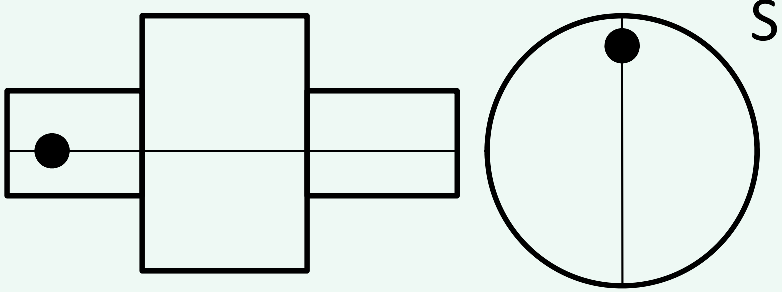
見やすくするため極表示を使う

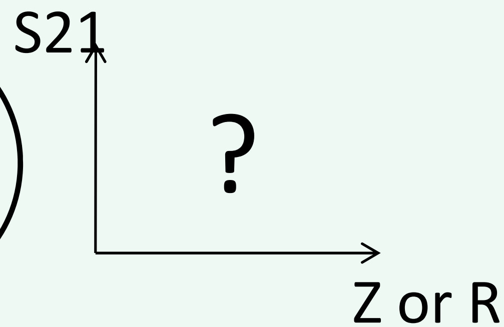
R・Z方向の分布を見る:
Bead Pull Method

①Input・Pickupアンテナを固定する
②見たい軸上に微小な金属体を一定速度で走らせる
③金属内に電磁場は入れないので共振周波数が乱れる
④同じ周波数を見ていればRF位相がずれる

$$\frac{\Delta U}{U_0} = \frac{\Delta f}{f_0} \qquad \Delta f = \frac{f_0}{2Q}\tan(-\Delta\varphi) \approx -\frac{f_0}{2Q}\Delta\varphi$$







真空中のMaxwell方程式

$$\mathrm{div}\vec{D} = 0 \qquad \vec{B} = \mu_0 \vec{H}$$

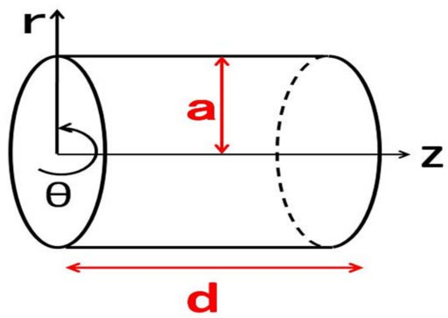
$$\mathrm{div}\vec{B} = 0 \quad \mathrm{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \quad \mathrm{rot}\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{0} \qquad \vec{D} = \varepsilon_0 \vec{E}$$

円筒形空洞共振器(pill box)への応用のまとめ

半径aの円筒に軸対称性を持つz方向に
進む平　面波(角周波数ω、波数β)

a=88.3mm,d=115.3mm
f=1300MHz;ILCの周波数

$$\begin{cases} E = E_0(x,y)e^{i(\omega t - kz)} \\ H = H_0(x,y)e^{i(\omega t - kz)} \end{cases}$$



Z成分とr、θ 成分に分け、Maxwell方程式に代入。

Et,Htは、**Ez,Hz**のみで決まる。

TM_{mnp}モード(加速モード):Hz=0,EZ≠0 の場合

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

:θ方向の節の数 **m**
:r方向の分布 **n**
:z方向の節の数 **p**

TE_{mnp}モード(キックモード):Hz≠0,Ez=0 の場合

$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a} r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

:θ方向の腹の数 **m**
:r方向の分布 **n**
:z方向の腹の数 **p**

高周波加速空洞のRFパラメーター

1. 空洞表面の発熱の評価
2. RFで電子の感じる実効的な電場の強さ
3. 空洞の形状の良さを表わすパラメータ
4. 加速電界の定義
5. 表面電界・表面磁場と加速電界の関係

常伝導体の表面抵抗と浸入深さ　その1

実際の金属は完全導体ではないため、表面抵抗が発生する。

Ampere-Maxwellの法則

$$\nabla\times H-\frac{\partial D}{\partial t}=J \;\Rightarrow\; \nabla\times\frac{\partial H}{\partial t}-\frac{\partial^2 D}{\partial t^2}=\frac{\partial J}{\partial t}=\sigma\frac{\partial E}{\partial t}\quad(\because J=\sigma E) \;\Rightarrow\; \nabla\times\left(-\frac{1}{\mu}\nabla\times E\right)-\frac{\partial^2 D}{\partial t^2}=\frac{\partial J}{\partial t}=\sigma\frac{\partial E}{\partial t}$$

オーム則

$$\Rightarrow\; \Delta E=\varepsilon\mu\frac{\partial^2 E}{\partial t^2}+\sigma\mu\frac{\partial E}{\partial t}$$

(電信方程式)

$$(\because \nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\Delta E)$$

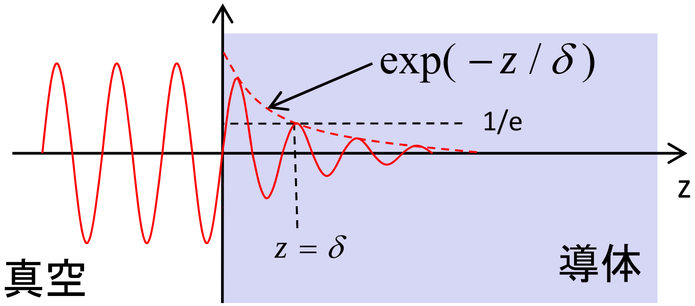
導体表面に向かい平面波を照射する場合を考える

平面波: $E_t(z,t)=E_0e^{-i(kz-\omega t)}$ 電信方程式に代入 $\Rightarrow$ $\dfrac{\partial^2 E_t}{\partial z^2}=\left(-\varepsilon\mu\omega^2+i\sigma\mu\omega\right)E_t$

$$\Rightarrow\; E_t=E_0\exp\left(\sqrt{-\varepsilon\mu\omega^2+i\sigma\mu\omega}\cdot z\right)=E_0\exp\left(\sqrt{\alpha+i\beta}\cdot z\right) \;\Rightarrow\; \begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\pm1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right)}$$

良導体の場合: $\dfrac{\sigma}{\varepsilon\omega}>>1$ 　　$E_t(z,t)=E_0\exp\left(-\sqrt{\dfrac{\mu\omega\sigma}{2}}z\right)\exp i\left(\omega t-\sqrt{\dfrac{\mu\omega\sigma}{2}}z\right)$

浸入深さ(Skin Depth): $\delta=\sqrt{\dfrac{2}{\mu\omega\sigma}}$



常伝導体の表面抵抗と浸入深さ　その2

浸入深さ(Skin Depth)： $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$

$$E_t(z,t) = E_0 \exp\left(-\frac{z}{\delta}\right)\exp i\left(\omega t - \frac{z}{\delta}\right) = E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp\left(i\omega t\right)$$

Faradayの法則　$\nabla \times E + \mu \dfrac{\partial H}{\partial t} = 0$　を考慮することで

$$H_t = \frac{1}{\mu}\int\left\{\frac{1}{\delta}(1+i)E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp(i\omega t)\right\}dt = \frac{1+i}{\mu\omega\delta}E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp(i\omega t)$$

Surface Impedance Z： $Z \equiv \dfrac{E_t}{H_t} \equiv R_s + iX_s = \mu\omega\sqrt{\dfrac{2}{\mu\sigma\omega}}\dfrac{1}{1+i} = \mu\omega\sqrt{\dfrac{2}{\mu\sigma\omega}}\dfrac{1-i}{2} = \sqrt{\dfrac{\mu\omega}{2\sigma}}(1-i)$

表面抵抗Rs:　$\mathrm{Re}\left(Z\right) = R_s = \sqrt{\dfrac{\mu\omega}{2\sigma}} = \dfrac{1}{\delta\sigma}$

課題⑥　20℃・1.3GHzでの、銅、アルミ、ニオブそれぞれの侵入深さ、表面抵抗を求めよ

電気伝導率σと電気抵抗率ρの関係： $\sigma = \dfrac{1}{\rho}$

銅 $:1.72 \times 10^{-8}[Ohm \cdot m]$　アルミ $:2.74 \times 10^{-8}[Ohm \cdot m]$　ニオブ $:1.61 \times 10^{-7}[Ohm \cdot m]$

Q値とは(共振スペクトル)

共振空洞内で表面抵抗により失われるエネルギー(壁面損失): $P_{loss} = \frac{1}{2} R_s \int |H|^2 dS$

他の考え方

空洞内に溜まったエネルギー(Stored Energy)の内単位時間に表面抵抗により失われるエネルギーは

エネルギー保存則より $P_{loss} = -\frac{dU}{dt}$ Stored Energyと壁面損失の比は一定 $Q = \frac{\omega U}{P_{loss}} = \frac{Const}{R_s} = \frac{\Gamma}{R_s}$

$$P_{loss} = -\frac{dU}{dt} = \frac{\omega U}{Q} \qquad \therefore U = U_0 e^{-\omega t/Q}$$

$U(t) = \frac{\varepsilon_0}{2}|E(t)|^2$ を考慮し $E(t) = E_0 e^{-\omega_0 t/2Q} e^{-i\omega t} = \int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega$

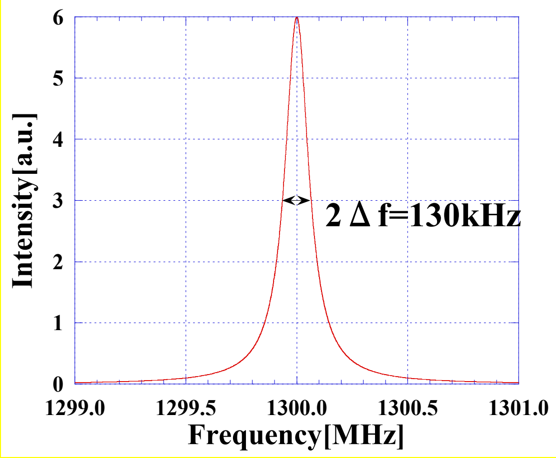
これをフーリエ逆変換すると $E(\omega) = \frac{1}{2\pi}\int_0^{\infty} E_0 e^{-\frac{\omega_0 t}{2Q}} e^{-i\omega_0 t} e^{i\omega t} dt = \frac{1}{2\pi}\int_0^{\infty} E_0 \exp(-\frac{\omega_0 t}{2Q} - i\omega_0 t + i\omega t) dt = \frac{1}{2\pi} \frac{E_0}{-\frac{\omega_0}{2Q} + i(\omega - \omega_0)}$

パワーで表すと、ローレンツの共鳴曲線になる

$$P = |E(\omega)|^2 = \left| \frac{A}{-\frac{\omega_0}{2Q} + i(\omega - \omega_0)} \right|^2 = \frac{A^2}{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0)^2}$$

$$Q = \frac{f_0}{2\Delta f} \qquad (\because \omega_0 = 2\pi f_0)$$

例) 共振周波数1300MHz、Q=10000の場合



加速空洞の性能を表わすパラメータ（Pillboxの場合） その1

加速電圧(Accelerating Voltage)：加速空洞を通過する際に粒子が受け取る実効的な加速電圧

ビームが空洞中心軸を光速で走るとき、粒子位置はz=ctと書ける。Pillbox TM010の場合 $k=\frac{p\pi}{d}=0$

$$V_c=\left|\int_0^{L_{eff}}E_0\exp\{i(\omega t-kz)\}dz\right|=\left|\int_0^{L_{eff}}E_0\exp\left\{i\frac{\omega}{c}z\right\}dz\right|=\left|\left[-\frac{iE_0}{\omega/c}\exp\left\{i\left(\frac{\omega}{c}\right)z\right\}\right]_0^d\right|=\frac{2E_0}{\omega/c}=\frac{2E_0d}{\pi}$$

$$\gamma^2=(\omega/c)^2-k^2\Rightarrow(\omega/c)^2=(\rho_{01}/a)^2=\left(\frac{\rho_{01}}{\rho_{01}d/\pi}\right)^2=\left(\frac{\pi}{d}\right)^2$$

$$\gamma=\rho_{01}/a\qquad a=\frac{\rho_{01}}{\pi}d$$

Transit-time factor ：時間変動する電磁場を粒子が通過する際、粒子が感じる時間効果

$$T=\frac{V_c}{\left|\int_0^d E_0dz\right|}=\frac{V_c}{E_0d}=\frac{2}{\pi}=\quad 0.64$$

加速電界(Effective accelerating gradient Eacc)：実際に空洞を電子が通過するときに感じる電界

$$E_{acc}=\frac{V_c}{L_{eff}}=\frac{dE_0T}{d}=E_0T=\frac{2E_0}{\pi}=\boxed{\quad}\,[MV/m]$$

Stored Energy ：空洞内部に蓄積したエネルギー

$$U=\frac{\mu_0}{2}\int|H|^2\,dV=\frac{\varepsilon_0}{2}\int|E|^2\,dV=\frac{\pi\varepsilon_0da^2E_0^2J_1^2(\rho_{01})}{2}=\boxed{\quad}\,[J]$$

Wall loss ：空洞表面でのエネルギー損失

$$P_{loss}=\frac{R_s}{2}\int|H_s|^2\,dS=\frac{R_s\pi\varepsilon_0a(a+d)E_0^2J_1^2(\rho_{01})}{\mu_0}=\boxed{\quad}\,[W]$$

課題⑦ それぞれの性能パラメータを計算してみよう

$E_0=1.0[MV/m]@1300MHz$

$\varepsilon_0=8.85418782\times10^{-12}[F/m]$

$\mu_0=1.2566370614\times10^{-6}[H/m]$

$c=2.99792458\times10^8[m/s]$

$R_s=9.4067\times10^{-3}[Ohm](20^OC,\text{Copper})$

$J_1(\rho_{01})=0.519$

$a=0.0883\,[m]\quad d=0.1153\,[m]$

加速空洞の性能を表わすパラメータ (Pillboxの場合) その2

Unloaded Quality factor:空洞自身のQ値

$$Q_0 = \frac{\omega U}{P_{loss}} = \frac{\omega \frac{\varepsilon_0}{2}\int_0^d |E|^2 dV}{\frac{R_s}{2}\int_S |H|^2 dS} = \frac{\omega \mu_0 da}{2R_S(a+d)} = \boxed{} \Rightarrow \quad Q_0 \propto 1/\sqrt{\omega}, \quad Q_0 \propto \sqrt{\sigma}$$

Geometrical factor:蓄積エネルギーと壁面損失の空洞形状に依存する比例定数

$$\Gamma = Q_0 \cdot R_S = \frac{\omega \mu_0 da}{2(a+d)} = \boxed{}\,[Ohm] \Rightarrow$$

Γはωに依存しない $\because d, a \propto \frac{1}{\omega}$

Shunt impedance:Stored Energyを空洞中心軸上にどれだけ集中できているかを示すパラメータ

$$R_{sh} = \frac{V_c^2}{P_{loss}} = \frac{\mu_0 \left(\frac{2d}{\pi}\right)^2}{R_S \pi \varepsilon_0 a(a+d) J_1^2(j_{01})} = \boxed{}\,[MOhm] \Rightarrow$$

形状と材料に依存する

$$R_{sh} \propto \frac{1}{\sqrt{\omega}}, \quad R_{sh} \propto \frac{1}{R_S}$$

R/Q(アール・バイ・キュー) :Shunt Impedanceのうち形状因子だけの定数

$$\frac{R_{sh}}{Q_0} = \frac{V_c^2}{P_{loss}}\frac{P_{loss}}{\omega U} = \frac{V_c^2}{\omega U} = \frac{8}{\pi^2 \varepsilon_0 c \rho_{01}^2 J_1^2(\rho_{01})} = \boxed{}\,[Ohm] \Rightarrow$$

周波数に依存しない

最大表面電場・最大表面磁場:加速電界との比で表わす

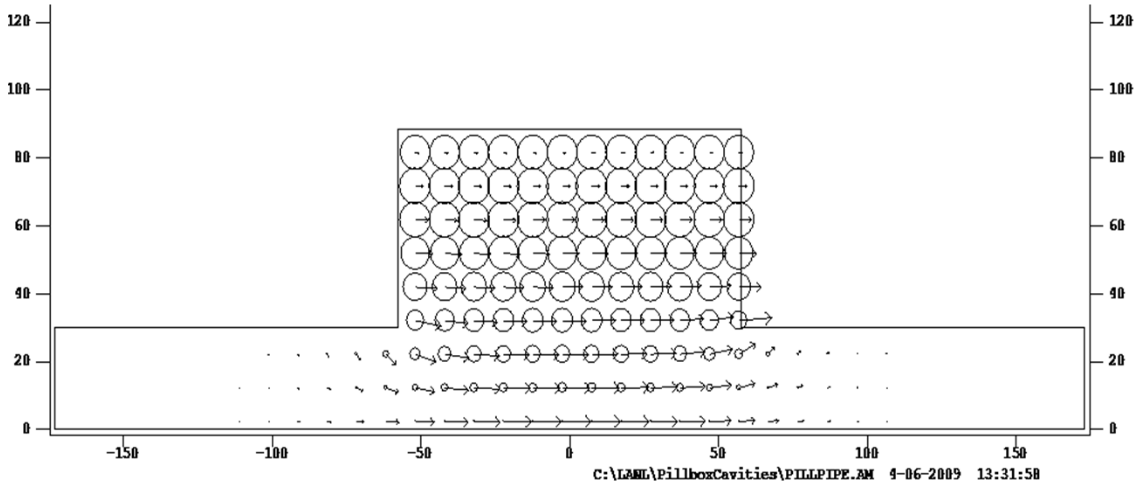
$$\frac{E_p}{E_{acc}} = \boxed{} \qquad \frac{H_p}{E_{acc}} = \frac{\pi}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} J_1\left(\frac{\rho_{01}}{a} r\Big|_{sp}\right) = \boxed{}\,[Oe/(MV/m)]$$

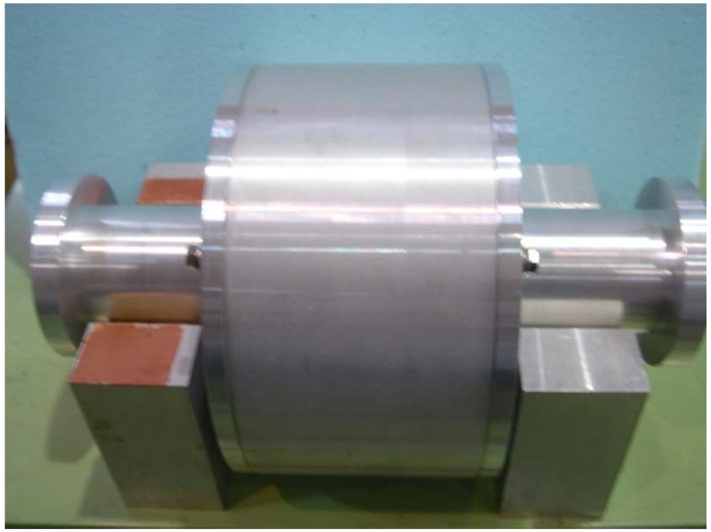
Nbの臨界磁場:1800[Oe]

$$1[Oe] = \frac{1}{4\pi} \times 10^3 [A/m]$$

課題⑧ Nb・Pillboxの場合の性能パラメータを計算してみよう

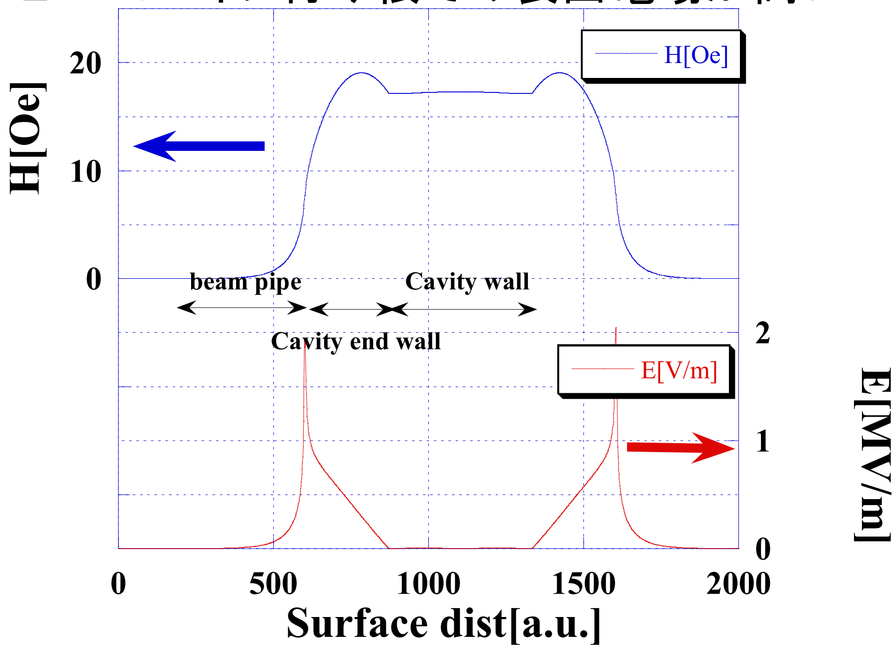
Φ60mm Beam Pipe付き空洞の設計





Bore diameter[mm]	60
Frequency[MHz]	1321
Transit-time factor	0.51
Geometrical factor[Ohm]	263.0
R/Q[Ohm]	126.8
Hp/Eacc[Oe/(MV/m)]	37.6
Ep/Eacc	4.04

Cavity壁面での表面磁場が高い
ビームパイプ付け根での表面電場が高い



Frequency dependence of the cavity RF parameters

Characteristics Parameter	Normal conducting ω dependence	Superconducting ω dependence
R_S [Ω]	$\omega^{1/2}$	ω^{2}
P_{loss} [W]	$\omega^{-3/2}$	No depend
U [J]	ω^{-3}	ω^{-3}
Q_O	$\omega^{-1/2}$	ω^{-2}
Rsh [Ω]	$\omega^{-1/2}$	ω^{-2}
Rsh/L [W/m]	$\omega^{-1/2}$	ω^{-1}
Γ [Ω]	No depend	No depend
R/Q [Ω]	No depend	No depend

単位長さ当たりのRshは、normal conductingでは $\sqrt{\omega}$ に比例して大きくなる。

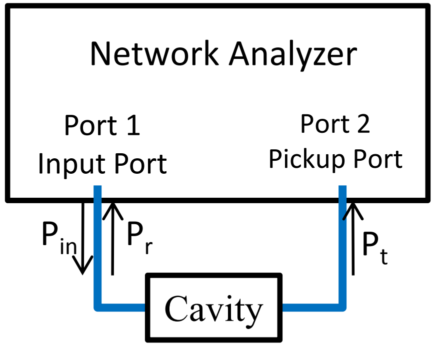
GLCがXーバンドを選択する理由。

Network Analyzerを用いたQ値の測定方法

測定可能なQ値：CavityとCableを含めた系全体のQ値(Loaded Q：Q_L)

$$Q_L = \frac{\omega U}{P_{loss} + P_t + P_e} = \frac{f_0}{2\Delta f}$$

空洞でのロス：P_{loss}
Input Portでのロス：P_e
Pickup Portでのロス：P_t

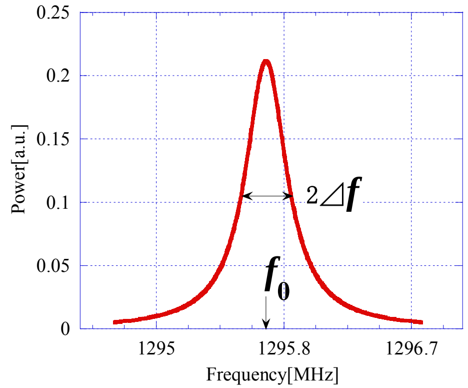


エネルギー保存則：

$$P_{loss} = P_{in} - P_t - P_r$$

Q_0は"Cavity Wall Loss"のみによるQ値、
Q_Lは"Cavity Wall Loss"＋"Cableから漏れ出す"パワーの合計によるQ値

Q_0を求めるにはWallロスとCableから漏れ出す
エネルギーの比率(Coupling)を求めればよい



Input Portから系全体を見たカップリング効率(Pickup Portのロスを含む)：β_{in}^*

$$\beta_{in}^* = \frac{P_e}{P_{loss}} = \frac{1 \pm \sqrt{\frac{P_r}{P_{im}}}}{1 \mp \sqrt{\frac{P_r}{P_{im}}}} = \frac{1 \pm S_{11}}{1 \mp S_{11}}$$

Over Coupling　：　$\beta_{in}^* > 1$
Under Coupling：　$\beta_{in}^* < 1$

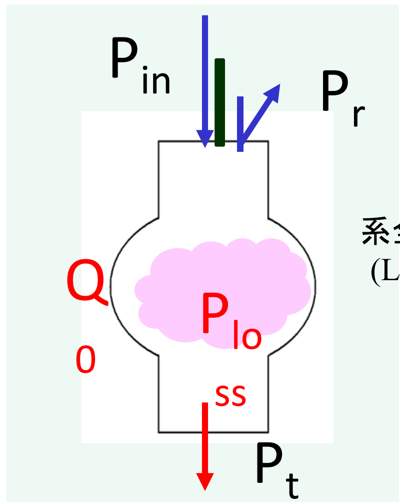
$$P_e = |S_{11}|^2 P_{in} = \left(\frac{1 - \beta_{in}^*}{1 + \beta_{in}^*} \right)^2$$

Pickup Portのカップリング：β_t

$$\beta_t = \frac{P_t}{P_{loss}}$$

空洞のみのQ値(Unloaded Q)

$$Q_0 = \left(1 + (1 + \beta_t)\beta^*_{in} + \beta_t\right) Q_L$$

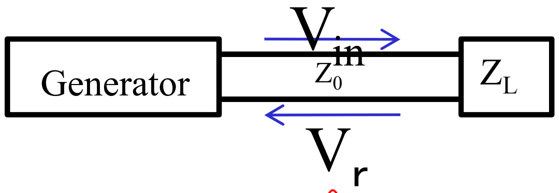


系全体のQ値
(Loaded Q_L)

カップリング

等価回路での考え方

例) インピーダンスZ_0のケーブルにZ_Lの物体を付けた場合の反射率

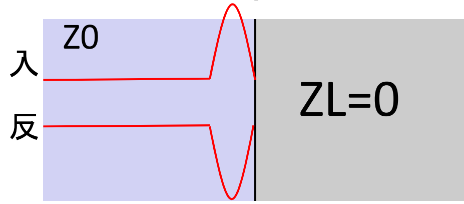


$$\Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

例) Z_LがShort終端(固定端)のとき

$$Z_L = 0 \Rightarrow \text{反射率} \quad \Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1$$

Short

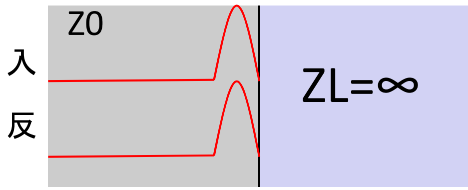


位相がπずれる

例) Z_LがOpen終端(自由端)のとき

$$Z_L = \infty \Rightarrow \Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\infty - Z_0}{\infty + Z_0} = 1$$

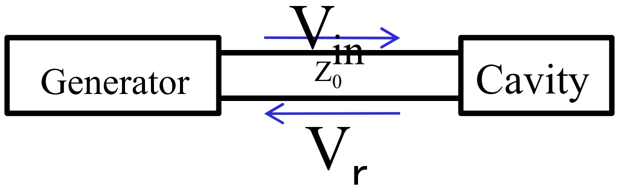
Open

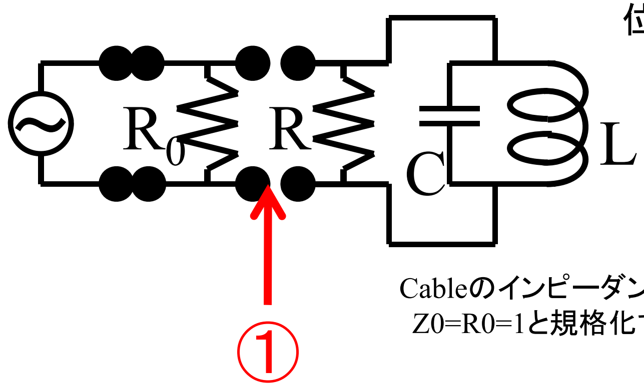


位相が変わらない

ここから空洞と伝送ラインのカップリングの話

例) 空洞共振器を等価回路で表す





スミスチャートの考え方

Cableのインピーダンスを
Z0=R0=1と規格化する

抵抗、コンデンサー、コイルのインピーダンス

$$Z_R = R \qquad Z_C = \frac{1}{i\omega C} \qquad Z_C = i\omega L$$

抵抗、コンデンサー、コイルを並列につないだとき、①から見たインピーダンスZ_Lは

$$\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + \frac{1}{1/i\omega C} + \frac{1}{i\omega L} = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right) = X + iY$$

1/ZL:アドミッタンス(インピーダンスの逆数)
X:コンダクタンス
Y:サセプタンス

カップリング

この場合の反射率

$$\Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L - 1}{Z_L + 1} = \frac{1 - 1/Z_L}{1 + 1/Z_L} = \frac{1 - (X + iY)}{1 + (X + iY)} = \frac{1 - X - iY}{1 + X + iY}$$

余談

インピーダンスZ_Lの実数成分をとると

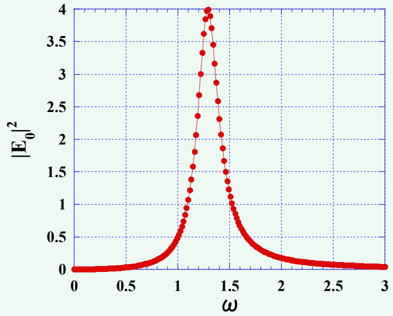
$$\mathrm{Re}\left(\frac{1}{Z_L}\right) = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

空洞に立つ電圧E_0は

$$\Rightarrow \quad E_0 = \frac{I_0}{Z_L} = \frac{I_0}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}}$$

$$|E_0|^2 = \frac{I_0{}^2}{(1/R)^2 + (\omega C - 1/\omega L)^2}$$

ローレンツカーブになっている



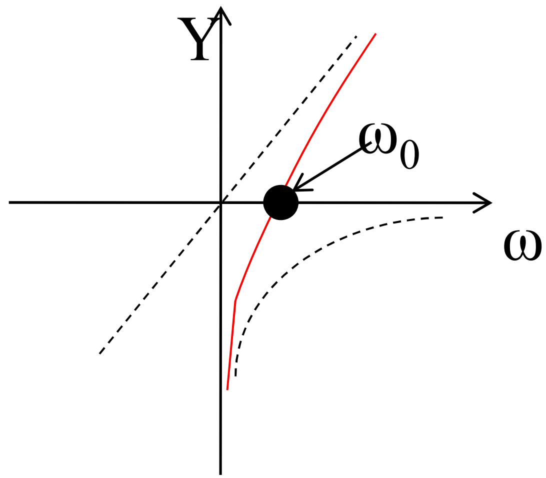
Yについて考える

$$Y = \omega C - \frac{1}{\omega L}$$

共振周波数から十分外れたとき

$$\omega \to \infty \Rightarrow Y = \infty \Rightarrow S_{11} = \frac{1 - X - iY}{1 + X + iY} \to -1$$

$$\omega \to 0 \Rightarrow Y = -\infty \Rightarrow S_{11} = \frac{1 - X - iY}{1 + X + iY} \to -1$$



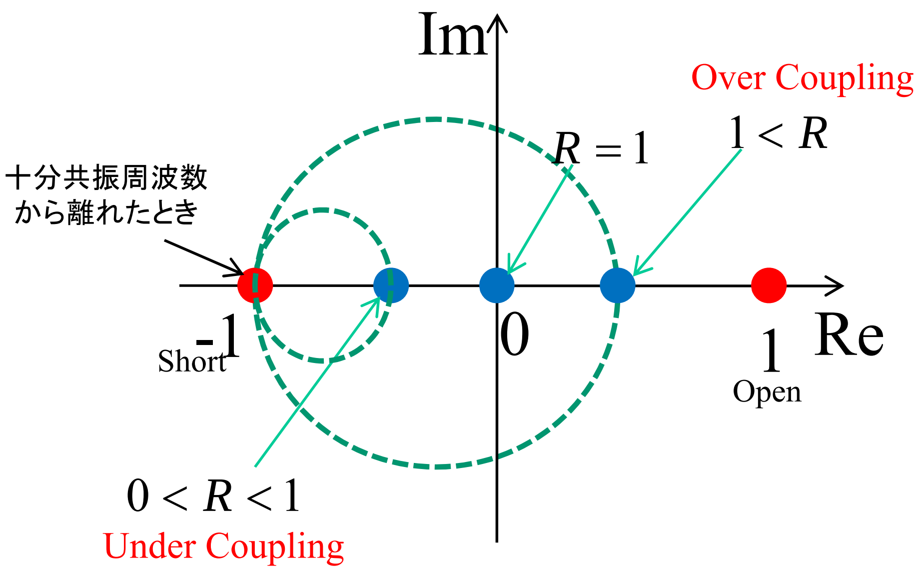
空洞の共振周波数ω0のとき

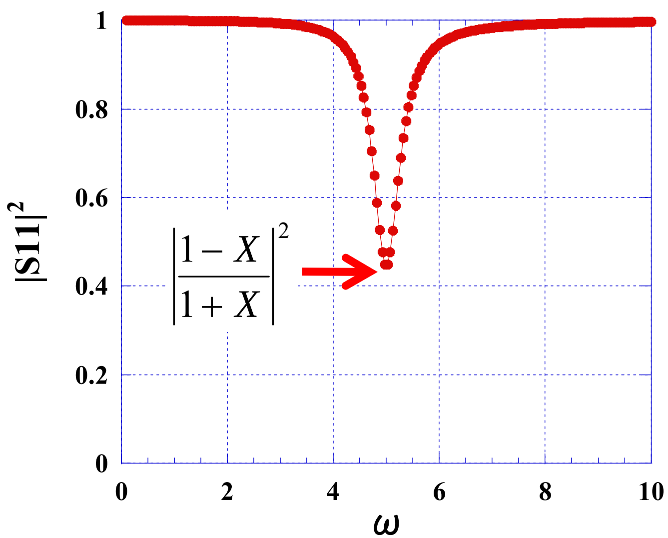
$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow Y = 0 \Rightarrow S_{11} = \frac{1 - X - iY}{1 + X + iY} \to \frac{1 - X}{1 + X}$$

カップリング

なので反射波は複素平面で考えると

$$|S_{11}|^2 = \frac{(1-X)^2 + (\omega C - 1/\omega L)^2}{(1+X)^2 + (\omega C - 1/\omega L)^2}$$





空洞の共振周波数ω0のとき

$$S_{11}(\omega_0) = \frac{1-X-iY}{1+X+iY} \rightarrow \frac{1-X}{1+X} = \frac{R-1}{R+1}$$

実数成分のみ

$0 < R < 1$ のとき $-1 < S_{11}(\omega_0) < 0 \longrightarrow$ Under

$R = 1$ のとき $S_{11}(\omega_0) = 0$

$1 < R$ のとき $0 < S_{11}(\omega_0) < 1 \longrightarrow$ Over

RはR0で規格化しており R0 はCableのインピーダンスを表す。R0をあらわに書くと

$$S_{11}(\omega_0) = \frac{R/R_0 - 1}{R/R_0 + 1} = \frac{\beta - 1}{\beta + 1}$$

この比率βを結合定数(Coupling)という

カップリング

なぜ円弧を描くのか、円弧を描くとして考える

$$S_{11}=\frac{1-X-iY}{1+X+iY}=\frac{(1-X-iY)(1+X-iY)}{(1+X)^2+Y^2}=\frac{(1-X)(1+X)-Y^2-iY(1+X+1-X)}{(1+X)^2+Y^2}$$

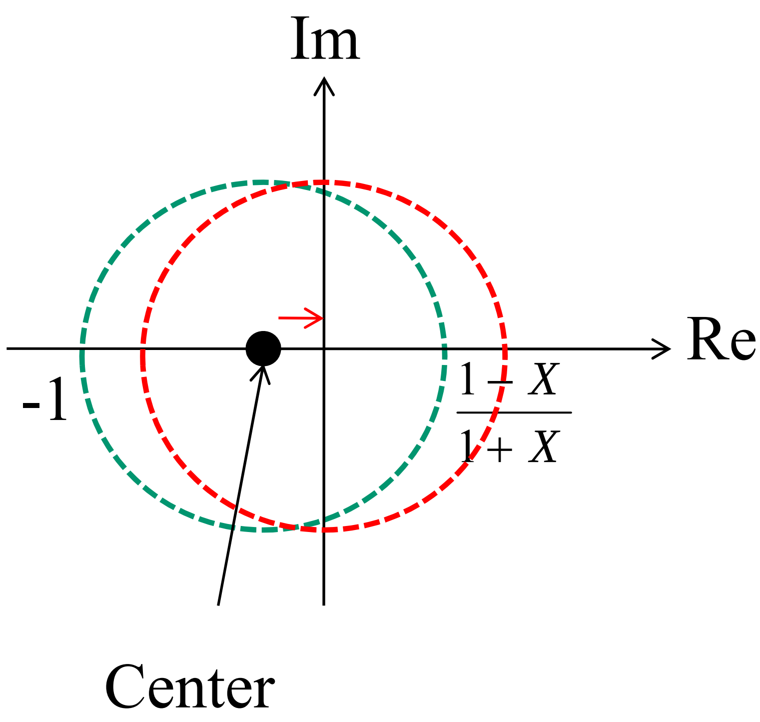
$$=\frac{1-X^2-Y^2-i2Y}{(1+X)^2+Y^2}$$

円の直径

$$\frac{1-X}{1+X}-(-1)=\frac{1-X+1+X}{1+X}=\boxed{\frac{2}{1+X}}$$

円の中心

$$-1+\frac{1}{1+X}=\frac{-1-X+1}{1+X}=\boxed{-\frac{X}{1+X}}$$



S11の円の中心はRe上にあるので中心を移す

$$\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)=\frac{1-X^2-Y^2}{(1+X)^2+Y^2}+\frac{X}{1+X}$$

$$=\frac{(1-X^2-Y^2)(1+X)+X\{(1+X)^2+Y^2\}}{\{(1+X)^2+Y^2\}(1+X)}$$

$$=\frac{(1-X^2-Y^2+X-X^3-XY^2)+(X+2X^2+X^3+XY^2)}{\{(1+X)^2+Y^2\}(1+X)}$$

カップリング

$$\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)=\frac{1+X^2+2X-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}=\frac{(1+X)^2-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}$$

$$\mathrm{Im}(S_{11})=\frac{-2Y}{(1+X)^2+Y^2}$$

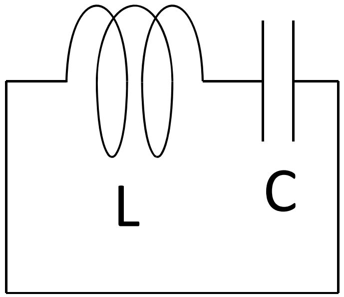
これが円を描くなら次の関係が成り立つ

$$\left[\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)\right]^2+\left[\mathrm{Im}(S_{11})\right]^2=\left(\frac{1}{1+X}\right)^2$$

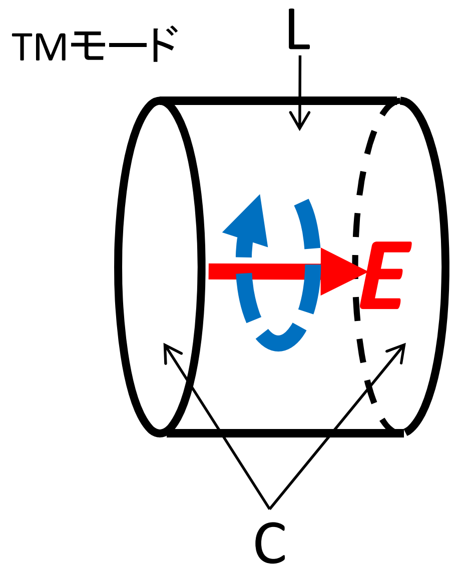
$$\left[\frac{(1+X)^2-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}\right]^2+\left[\frac{-2Y}{(1+X)^2+Y^2}\right]^2=\left(\frac{1}{1+X}\right)^2$$

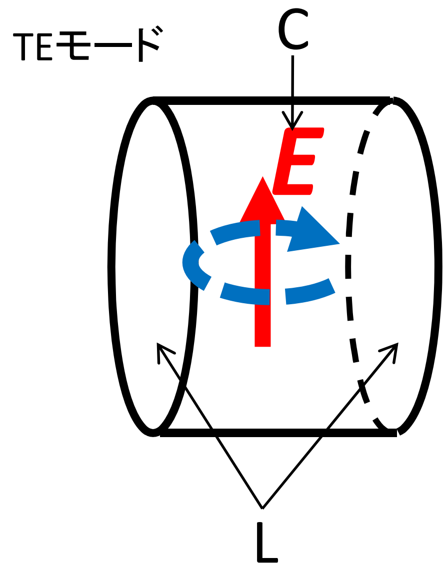
$$\Rightarrow\ \frac{\left\{(1+X)^2-Y^2\right\}^2+4Y^2(1+X)^2}{\left\{(1+X)^2+Y^2\right\}^2(1+X)^2}=\frac{\left\{(1+X)^2+Y^2\right\}^2}{\left\{(1+X)^2+Y^2\right\}^2(1+X)^2}=\frac{1}{(1+X)^2}$$

Ｙの値によらず円軌道上を動くことが分かる！

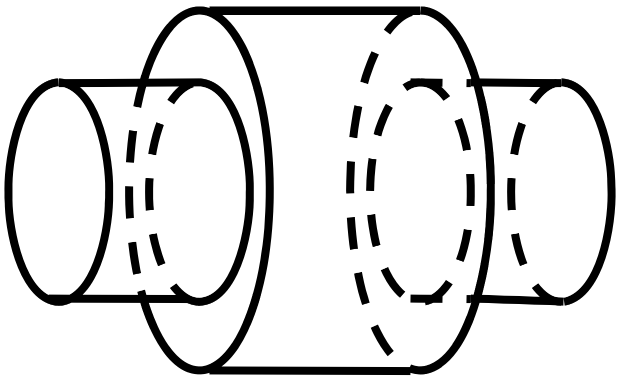


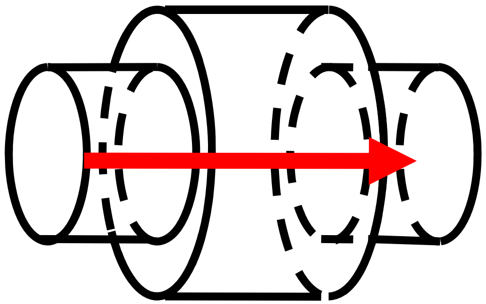
$$f = \frac{1}{2\pi\sqrt{LC}}$$

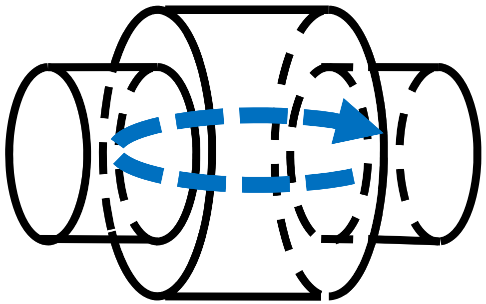




Beam Pipeが付くと







TM Modeの場合
C減少⇒ f 増加

TE Modeの場合
L増加⇒ f 減少

4.2 Criteria General for Cavity Shape

- Suppressed Multipacting
- Lower Surface Electric field

2001-2004

- Lower Surface Magnetic field
- High Efficient
- High Gradient

$\Big\}$ incorporate

Real Cavity Design

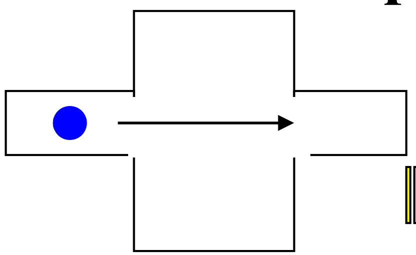
Diameter of BP → bigger
Ep/Eacc → larger
Hp/Eacc → larger
R/Q → larger
Cell to cell coupling → smaller
HOM issue → more serious

1) Need a hole on the cavity
   for electron to pass the cavity
2) Need RF input port
3) HOM coupler port

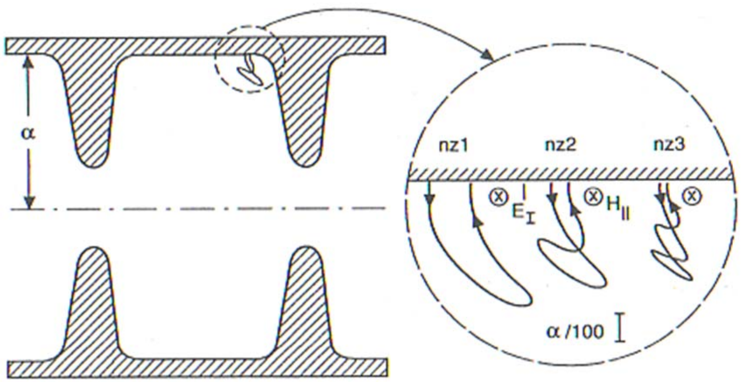
Need Beam pipes on both Ends

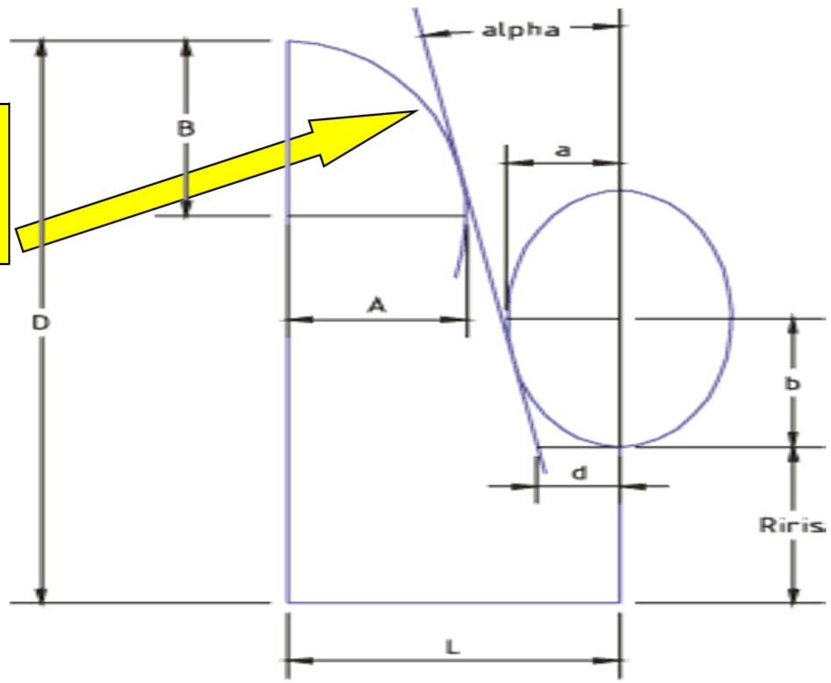
4) High efficient cavity
5) Better performance
   Smaller Ep/Eacc : Field emission
   Smaller Hp/Eacc : Multipaction

Optimization of cell shape
Multi-cell cavity

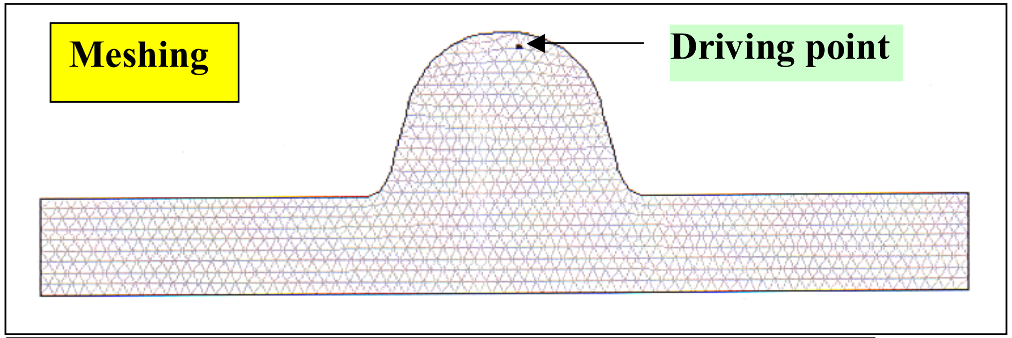


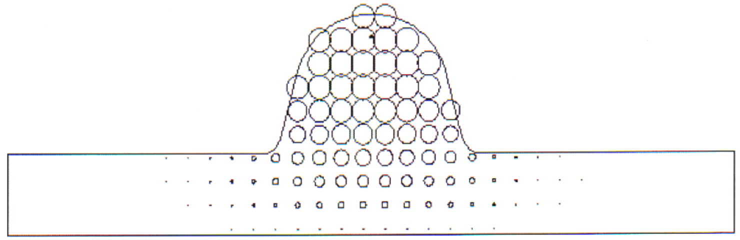
Choose spherical shape
to reduce multipacting



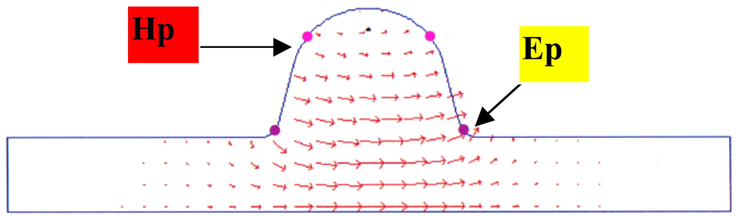


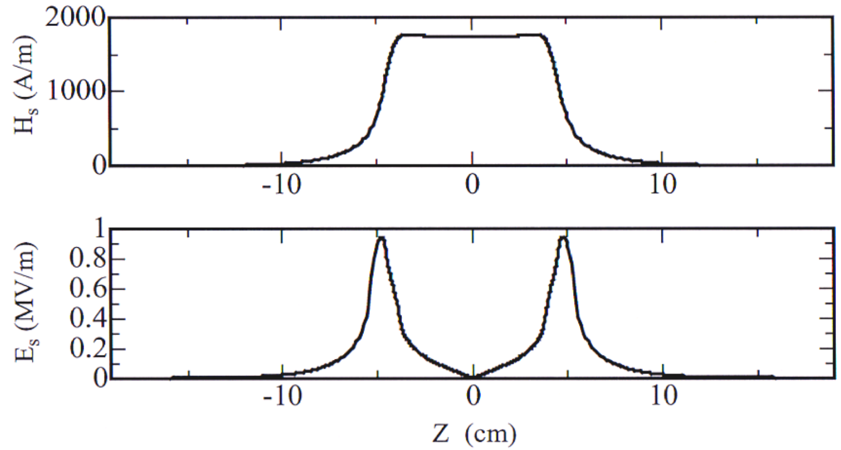
Real cavity design





All calculated values below refer to the mesh geometry only.
Field normalization (NORM = 0): EZERO = 1.00000 MV/m
Length used for E0 normalization = 10.76000 cm
Frequency (starting value = 1300.000) = 1293.77430 MHz
Particle rest mass energy = 0.510999 MeV
Beta = 1.0000000
Normalization factor for E0 = 1.000 MV/m = 7048.913
Transit-time factor Abs(T+iS) = 0.5454664
Stored energy = 0.0038869 Joules
Using standard room-temperature copper.
Surface resistance = 9.38405 milliOhm
Normal-conductor resistivity = 1.72410 microOhm-cm
Operating temperature = 20.0000 C
Power dissipation = 1118.1551 W
Q = 28257.6 Shunt impedance = 96.230 MOhm/m
Rs*Q = 265.171 Ohm Z*T*T = 28.632 MOhm/m
r/Q = 109.024 Ohm Wake loss parameter = 0.22157 V/pC
Average magnetic field on the outer wall = 1729.9 A/m, 1.40411 W/cm^2
Maximum H (at Z,R = 3.32643,8.55466) = 1753.44 A/m, 1.44258 W/cm^2
Maximum E (at Z,R = 4.75232,4.24425) = 0.946176 MV/m, 0.02953 Kilp.
Ratio of peak fields Bmax/Emax = 2.3288 mT/(MV/m)
Peak-to-average ratio Emax/E0 = 0.9462





Example of calculation of final characteristic cavity parameters

In case of single cell cavity with beam tube, definition acceleration field becomes more questionable due to the field leaking into the tube.

We define the effective cavity length as βλ/2 no concerning to the real cell length.

Superfish outputs

f=1293.77430 MHz

Ploss=1118.1551 W

Rs･Q=265.171 Ohm

Qo=28257.6

(Rsh/Q)=109.24 Ohm

Hs,max=1753.44 A/m

Es,max=0.946176 MV/m

演習問題

左のパラメターから以下の空洞パラメターを計算せよ。

Rsh =

$V =$

λ =

Eacc = V/Leff 但し、有効加速長をλ/2 と定義する。

Hp/Eacc = **Oe/(MV/m)** 但し、**1[A/m]** = $4\pi\times10^{-3}$ **[Oe]**

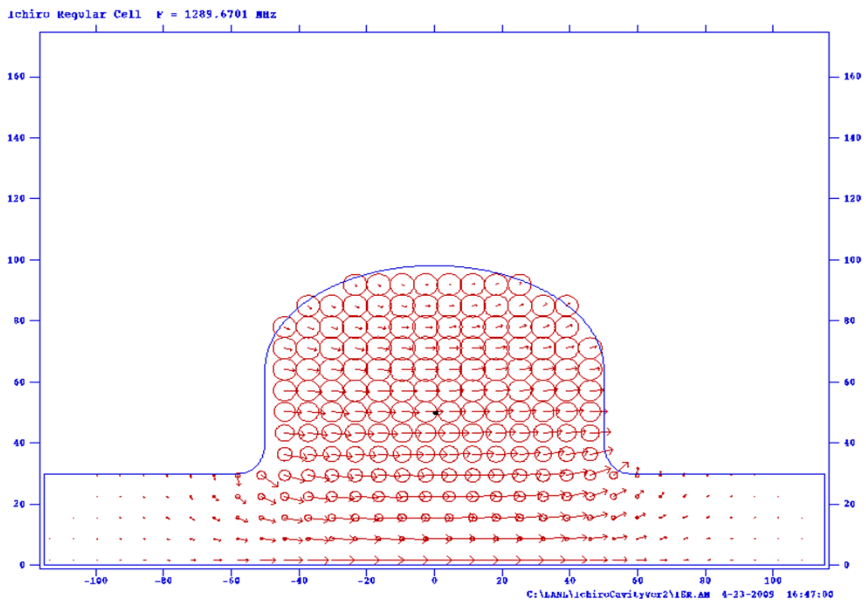
Ep/Eacc =

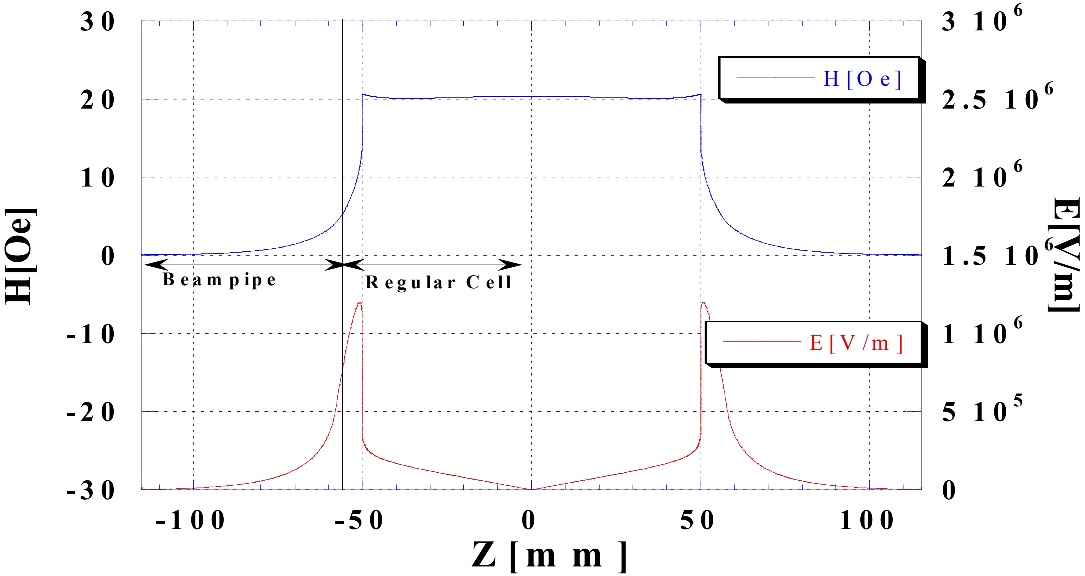
Eacc $= C\cdot\sqrt{P_{loss}\cdot Q_O}$,

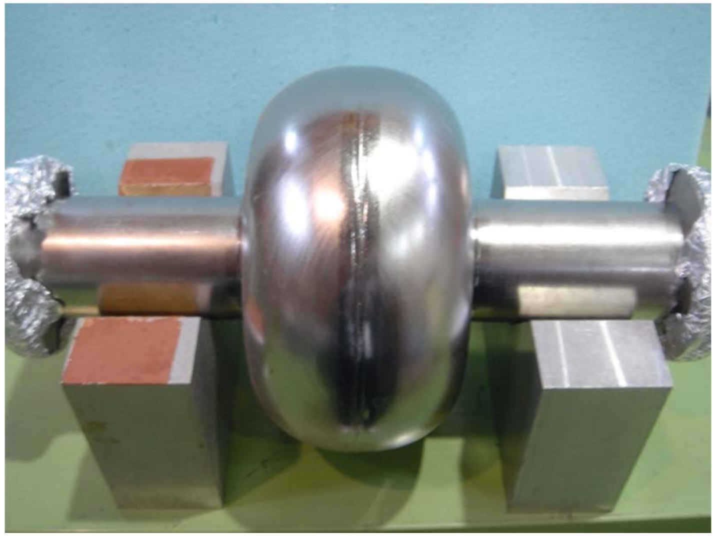
C =

Ichiro Single Cell(Regular Cell)空洞の設計

1.3GHzの共振周波数を持ち、表面電磁場を抑制する構造を作る。



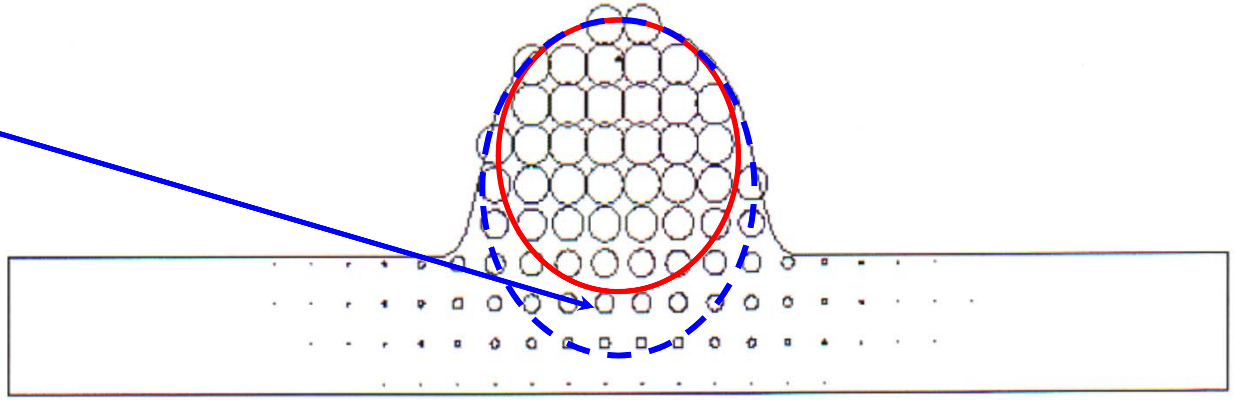




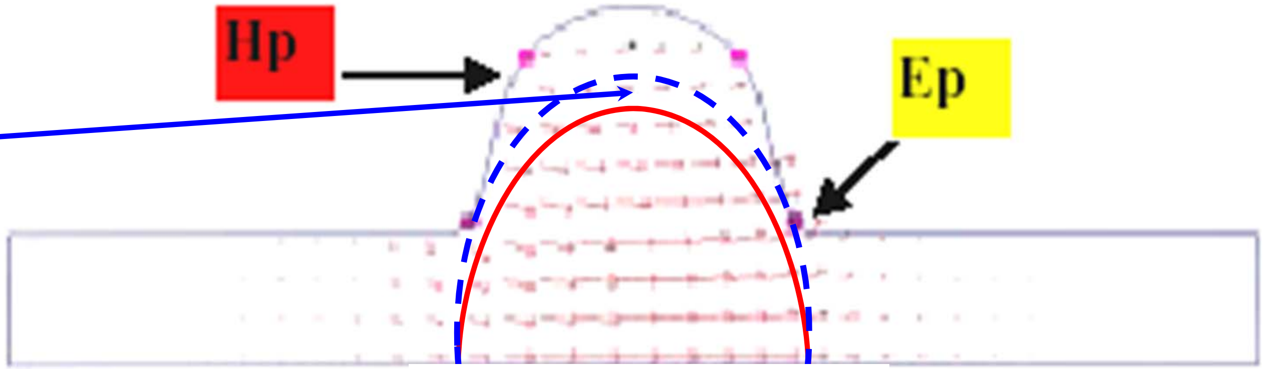
Ichiro Regular Cell	
Frequency[MHz]	1289.6701
Transit-time factor	0.5803802
Stored energy[J]	0.0039884
Surface resistance[nOhm]	26.5302
Wall loss[μW]	3042.4839
Unloaded Q	1.06E+10
Geometrical factor[Ohm]	281.819
R/Q[Ohm]	138.566
Shunt impedance[GOhm]	1471.9709
Hp[A/m]	20.644933
Hp/Eacc[Oe/(MV/m)]	35.571394
Ep/Eacc	2.0718832
EaccMax	50.602459

空洞のRF特性の直感的把握

BP → 小
Hの領域 → 大
Hp/Eacc → 小



BP → 小
Eの領域 → 大
Ep/Eacc → 小



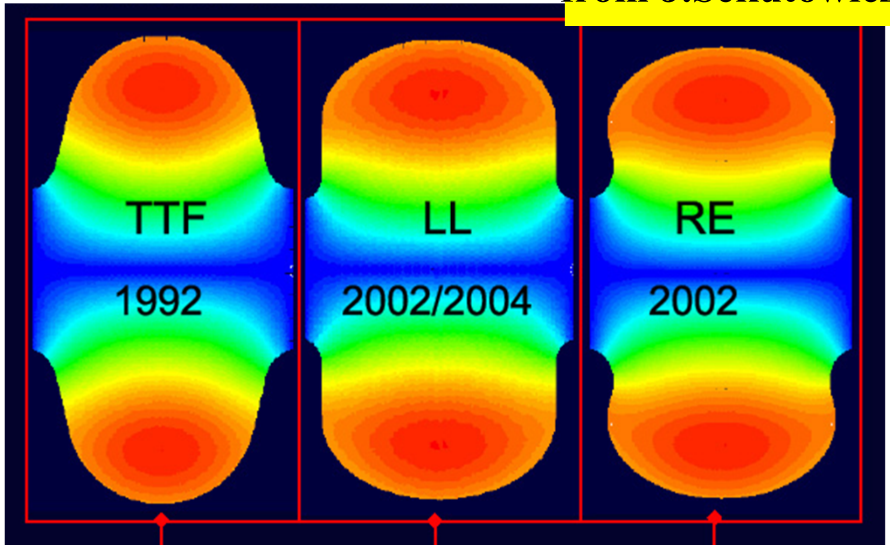
空洞設計では、beam pipeの径が大きな影響を与える。その径が小さくなるとHp/Eacc, Ep/Eacc,が小さくなる。また、beam 軸にE-fieldが集中するのでシャントイピーダンスが上がる。ただし、多連空洞ではcell-to-cell couplingが小さくなり、fieldの安定度が得られにくい。
また、HOMも問題になる。これらの全体的考察から最適化される。

High Gradient Shapes

Cavity shape designs with low Hp/Eacc

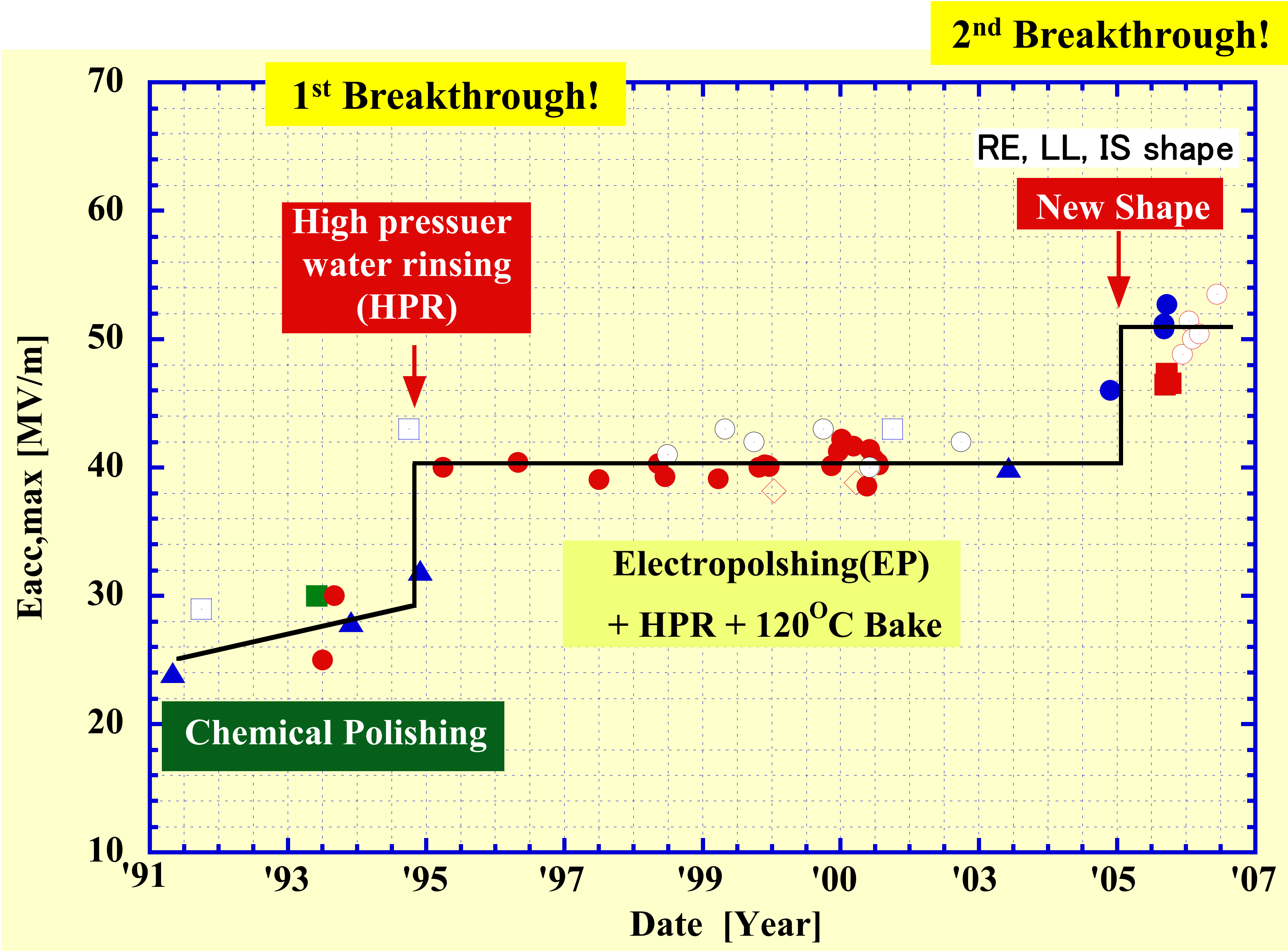
from J.Sekutowicz lecture Note

TTF: TESLA shape
Reentrant (RE): Cornell Univ.
Low Loss(LL): JLAB/DESY
Ichiro-Single (IS): KEK

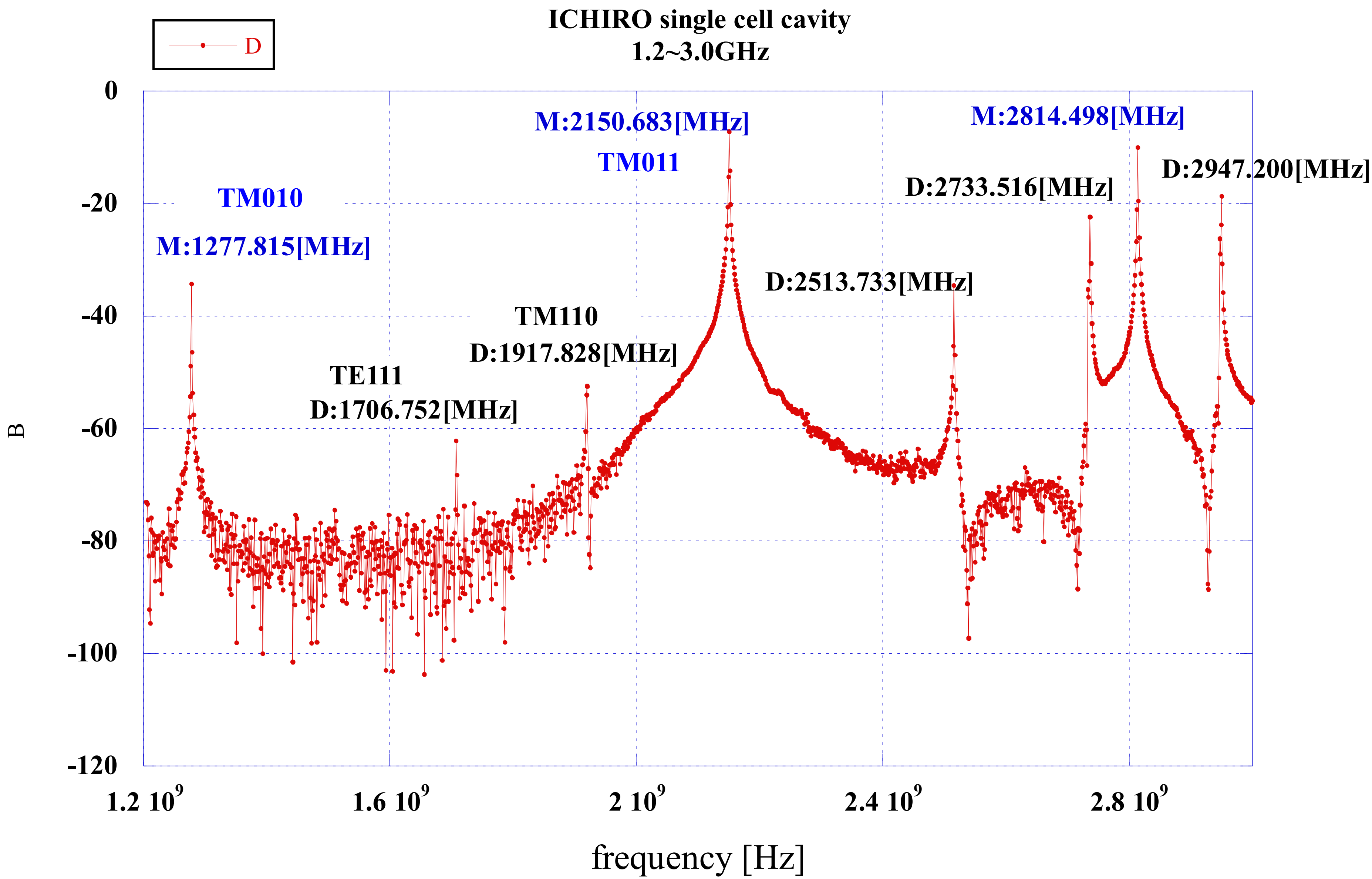


	TESLA	LL	RE	IS
Diameter [mm]	**70**	**60**	**66**	**61**
Ep/Eacc	**2.0**	**2.36**	**2.21**	**2.02**
Hp/Eacc [Oe/MV/m]	**42.6**	**36.1**	**37.6**	**35.6**
R/Q [W]	**113.8**	**133.7**	**126.8**	**138**
G[W]	**271**	**284**	**277**	**285**
Eacc max	**41.1**	**48.5**	**46.5**	**49.2**

Eacc vs. Year

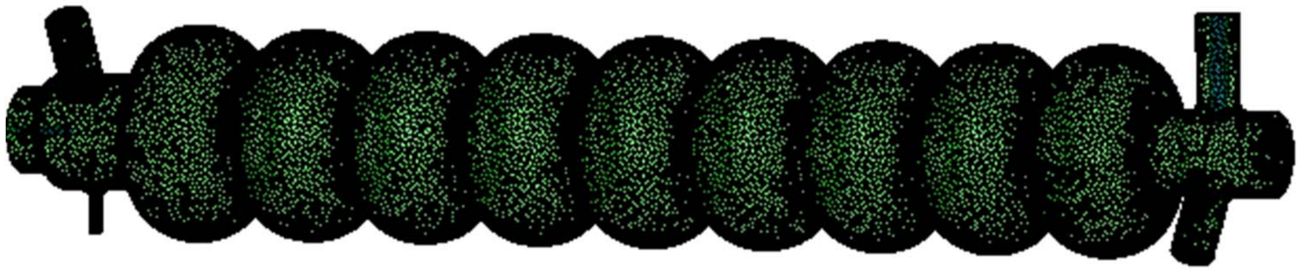

2nd Breakthrough!
1st Breakthrough!
RE, LL, IS shape
New Shape
High pressuer water rinsing (HPR)
Electropolshing(EP) + HPR + 120^{O}C Bake
Chemical Polishing
Eacc,max [MV/m]
70
60
50
40
30
20
10
'91
'93
'95
'97
'99
'00
'03
'05
'07
Date [Year]

HOM Modes in Single Cell Cavity



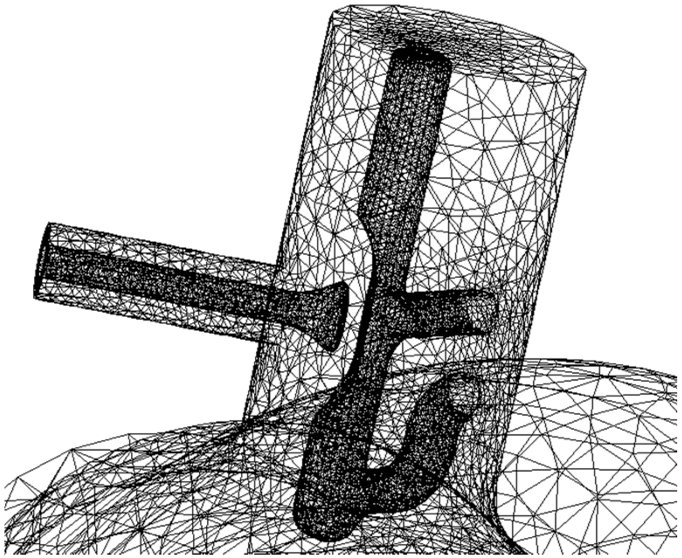
4.3 Multi-Cell Structure RF Design

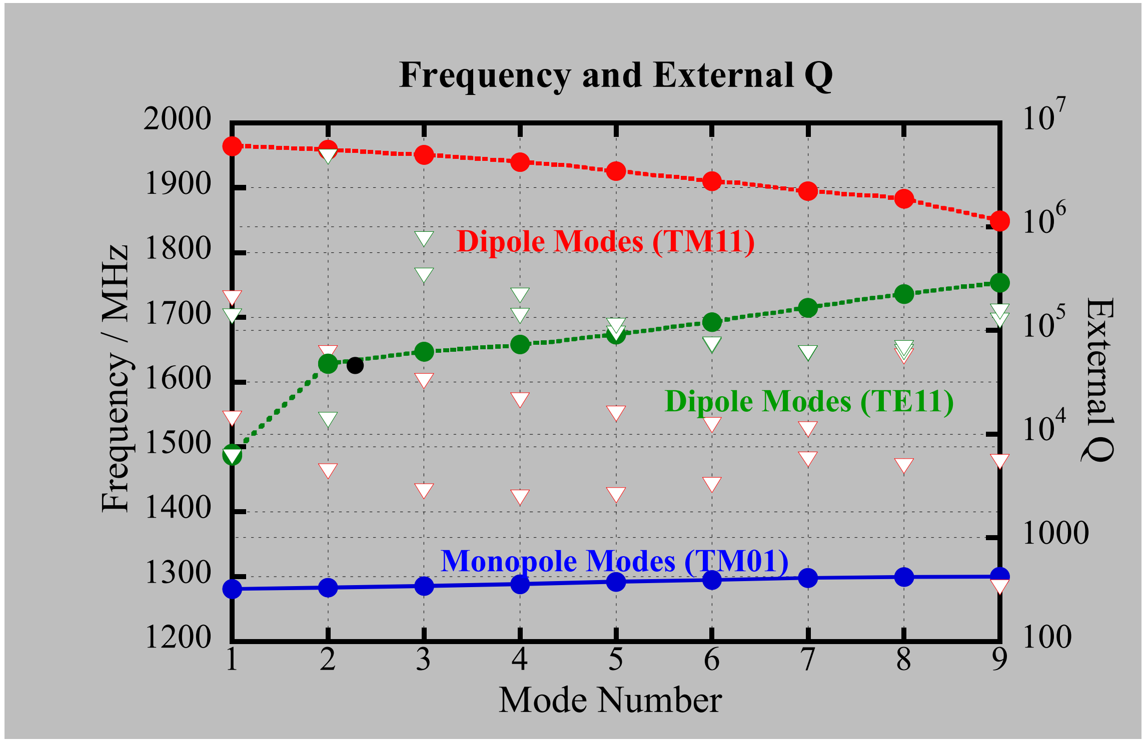
Element Model (Mesh)



Simulation of Higher Order Mode Damping

Mesh of HOM Damper





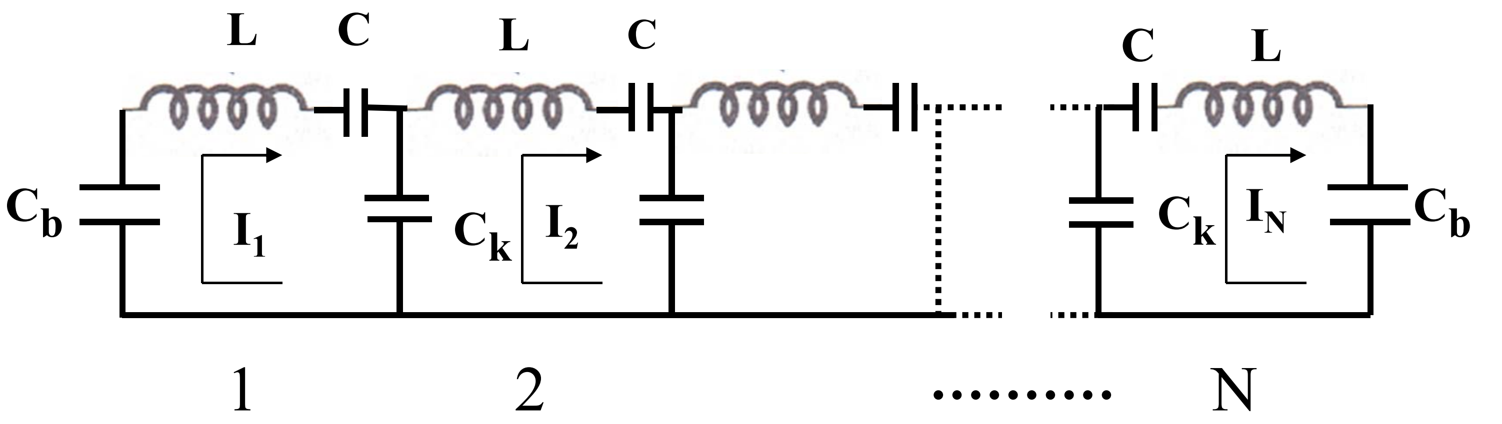
Criteria for Multi-cell Structures

Pros and cons for a multi-cell structure

- *Cost of accelerators is lower (less auxiliaries: LHe vessels, tuners, fundamental power couplers, control electronics)*
- *Higher real-estate gradient (better fill factor)*

- *Field flatness vs. N*
- *HOM trapping vs. N*
- *Power capability of fundamental power couplers vs. N*
- *Chemical treatment and final preparation become more complicated*
- *The worst performing cell limits whole multi-cell structure*

Equivalent Circuit Model for Multi-Cell Cavity



$$\left(\frac{1}{i\omega C_b}+i\omega L\right)I_1+\left(\frac{1}{i\omega C}\right)I_1+\left(\frac{1}{i\omega C_k}\right)(I_1-I_2)=0$$

$$\frac{1}{i\omega C_k}(I_j-I_{j-1})+\left(i\omega L+\frac{1}{i\omega C}\right)I_j+\left(\frac{1}{i\omega C_k}\right)(I_j-I_{j+1})=0$$

$$1<j<N$$

$$\left(\frac{1}{i\omega C_k}\right)(I_N-I_{N-1})+\left(i\omega L+\frac{1}{i\omega C}\right)I_N+\left(\frac{1}{i\omega C_b}\right)I_N=0$$

$$(1+k+\gamma)I_1-kI_2=\Omega I_1$$

$$-kI_{j-1}+(1+2k)I_j-kI_{j+1}=\Omega I_j,\quad 1<j<N$$

$$-kI_{N-1}+(1+k+\gamma)I_N=\Omega I_N$$

$$\omega_0^2=\frac{1}{LC},\ k=\frac{C}{C_k},\ \gamma=\frac{C}{C_b},\ \text{and } \Omega=\frac{\omega^2}{\omega_0^2}$$

$$\begin{bmatrix} 1+k+\gamma & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & -k & 1+2k & -k \\ 0 & \cdots & 0 & -k & 1+k+\gamma \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}=\Omega\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}$$

Solution for j-th cell in m-th mode

π-mode

$$\mathrm{v} = \frac{1}{\sqrt{N}}[1,\ \text{-}1,\ 1,\ \text{-}1,\ \cdots]$$

$$\begin{bmatrix} 1+k+\gamma & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & -k & 1+2k & & -k \\ 0 & \cdots & 0 & -k & 1+k+\gamma \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \\ \end{bmatrix} = \Omega^{(N)} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \\ \end{bmatrix} \Rightarrow$$

General equation

$$\begin{bmatrix} 1+3k & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & -k & 1+2k & & -k \\ 0 & \cdots & 0 & -k & 1+3k \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix} = \Omega \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}$$

$$1+2k+\gamma = \Omega^{(N)}$$

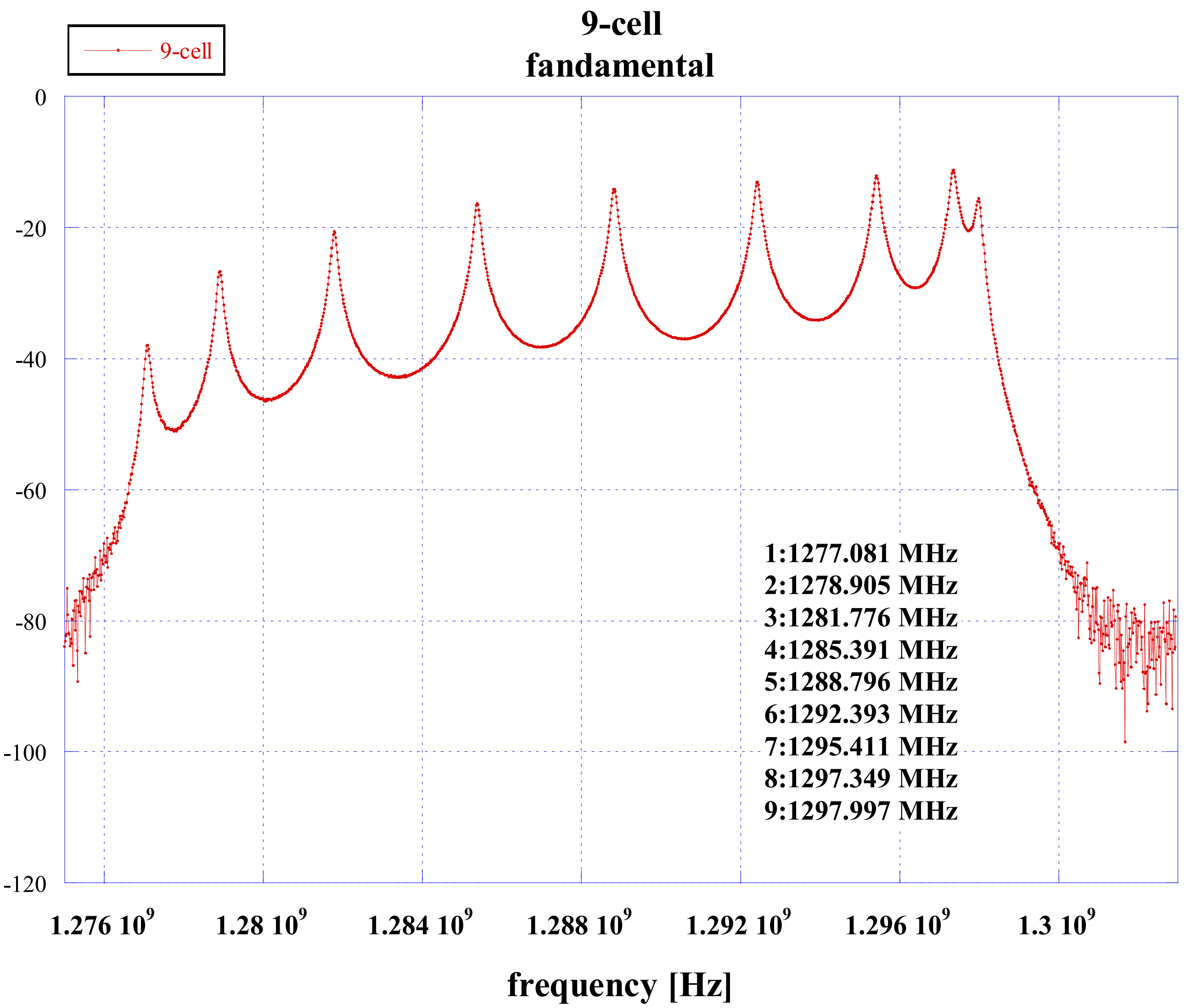
$$-1-4k = -\Omega^{(N)} \rightarrow \quad \gamma = 2k$$

$$v_j^{(m)} = B^{(m)} \sin\left[m\pi \left(\frac{2j-1}{2N} \right) \right], \quad B^{(m)} = \sqrt{\frac{2-\delta_{m,N}}{N}}$$

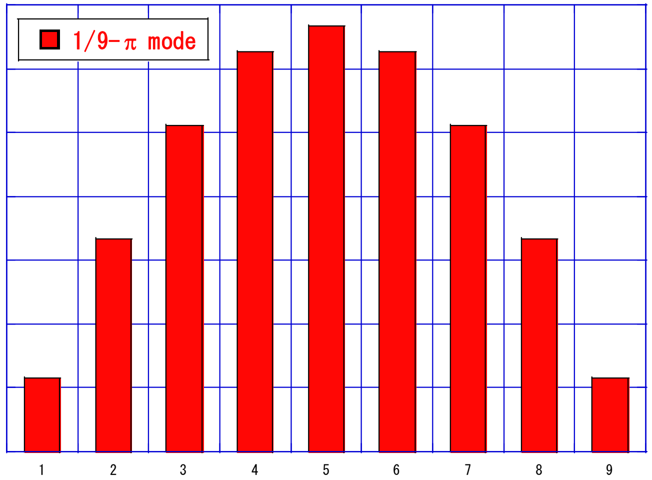
m = m-th mode,

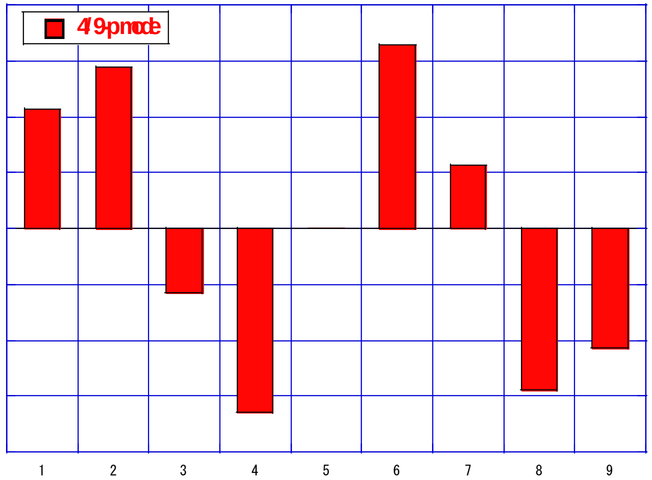
j = j-th cell in mode m-th

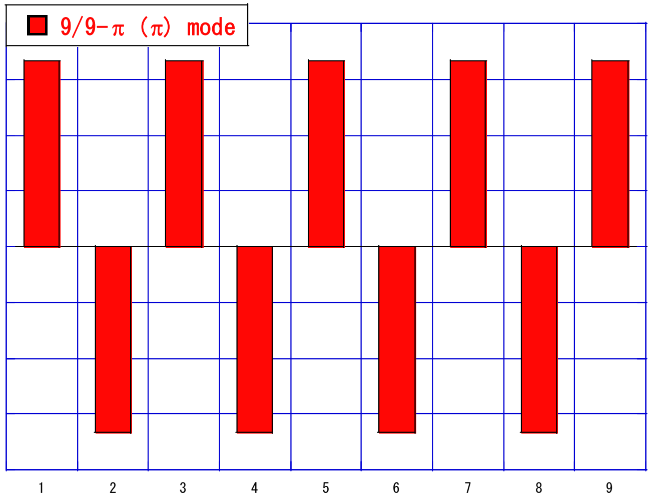
加速モードにおけるパスバンド



Field distribution in pass-bands

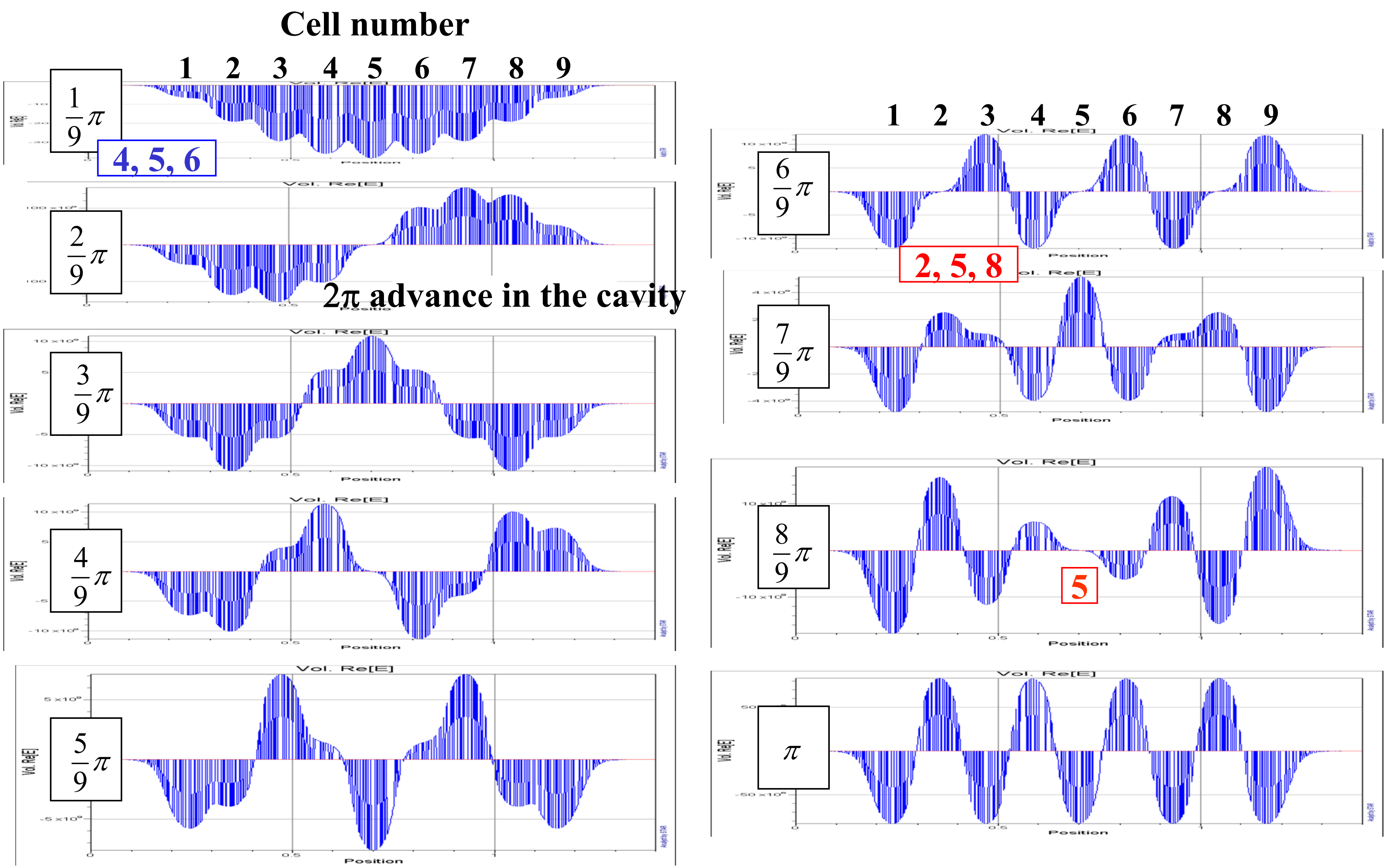




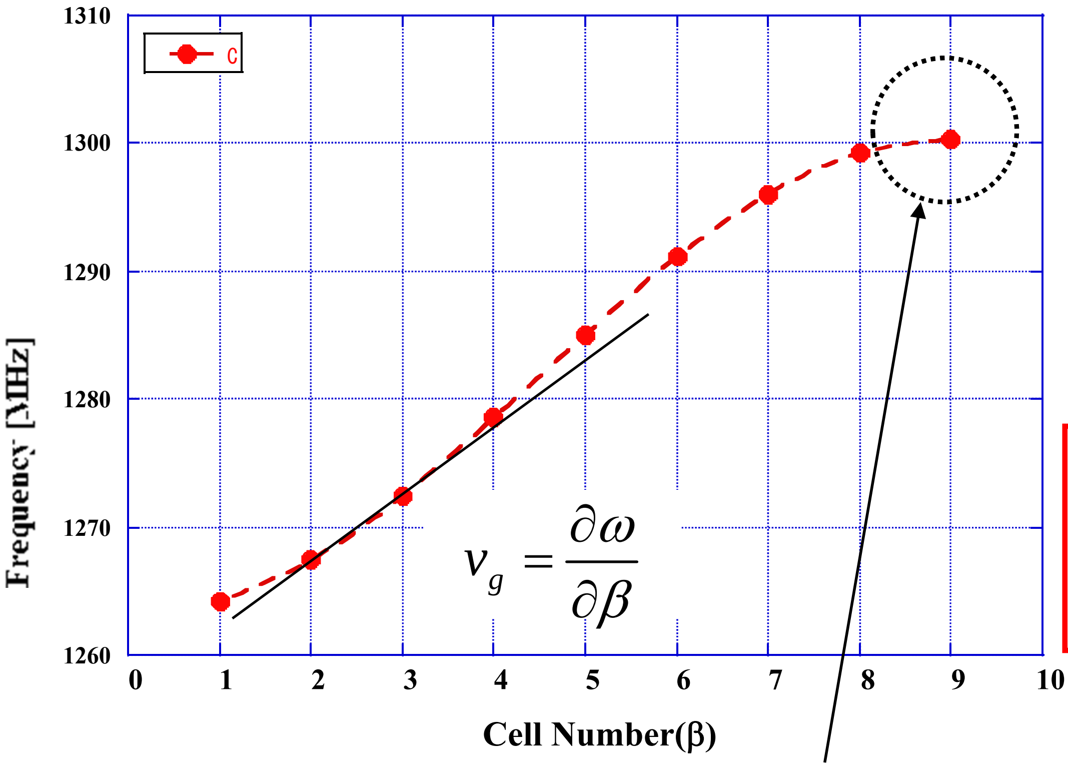


1空洞当たり mπ のphase advance

Pass-band modes of TM010 in a 9-cell cavity



Pass-band modes frequency



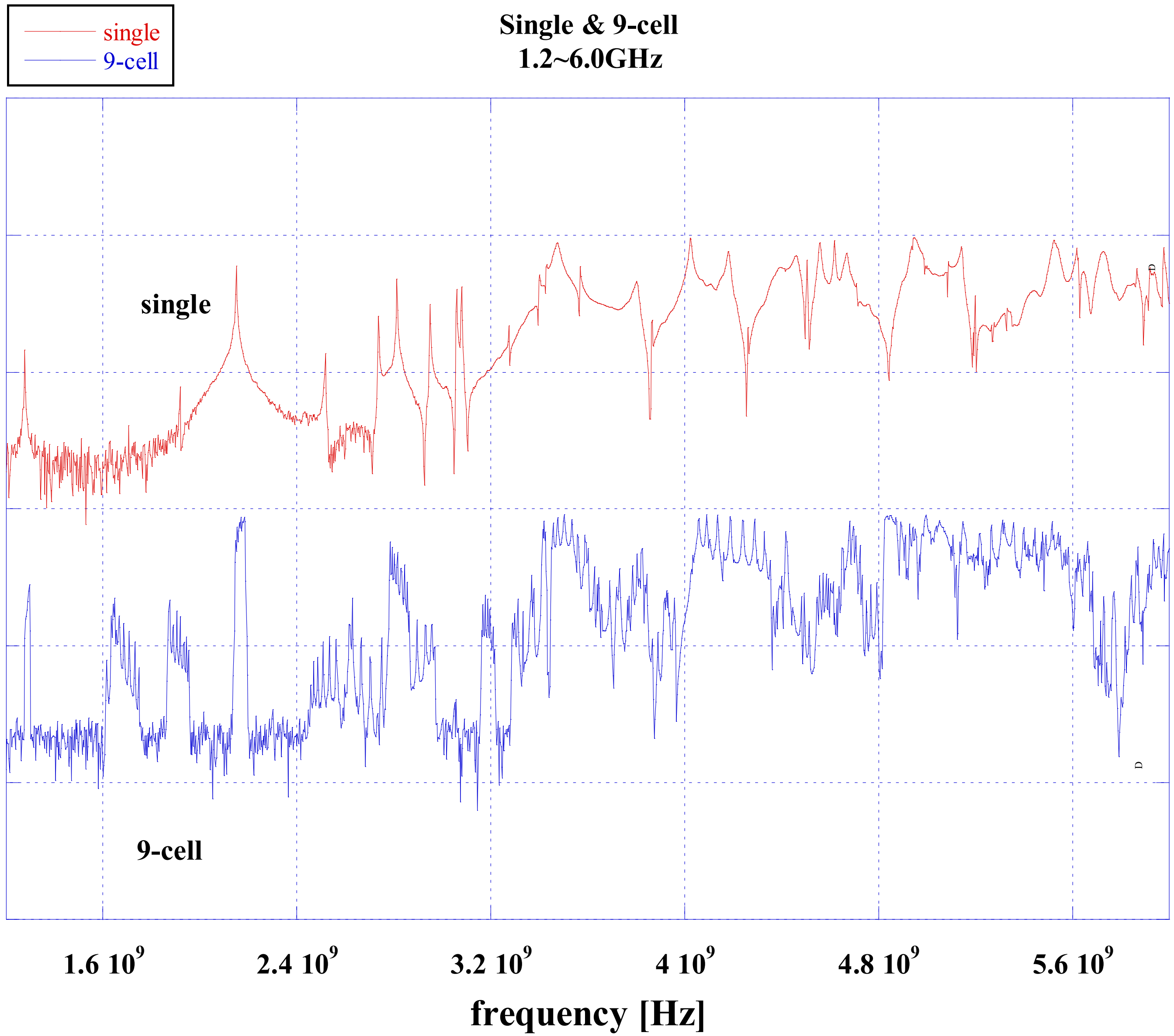
$$v_g = \frac{\partial \omega}{\partial \beta}$$

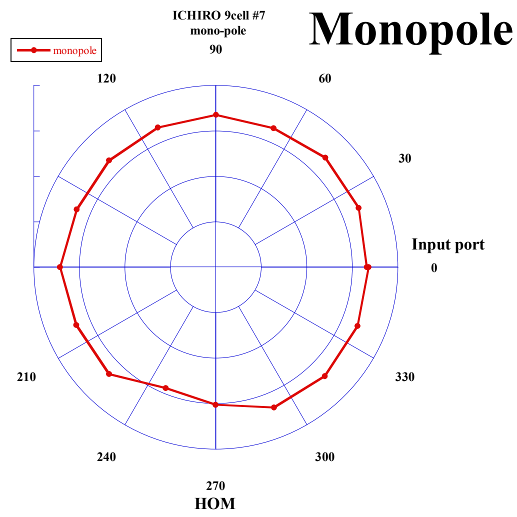
$$\begin{aligned} v_{j-1}^{(m)} + v_{j+1}^{(m)} &= B^{(m)} \sin\left[\left(m\pi \frac{(2j-1)}{2N}\right) - \frac{m\pi}{N}\right] + \\ &\quad B^{(m)} \sin\left[\left(m\pi \frac{(2j-1)}{2N}\right) + \frac{m\pi}{N}\right] \\ &= 2 v_j^{(m)} \cos\left(\frac{m\pi}{N}\right) \end{aligned}$$

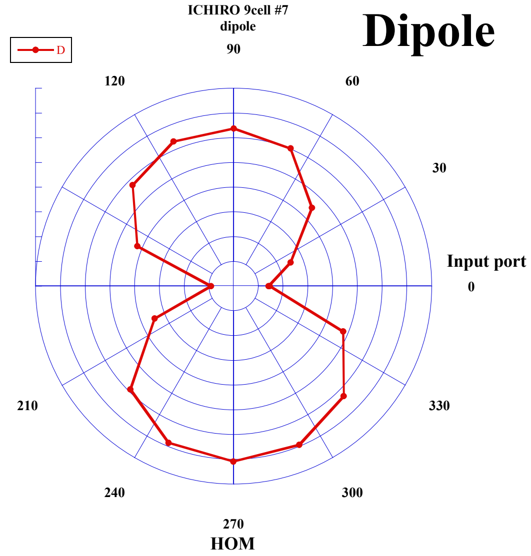
$$\boxed{\Omega^{(m)} = \left(\frac{f_m}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{m\pi}{N}\right)\right]}$$

セル数が無限大になると　v_g(p-mode)=0　になる。RFの流れがなくなる。
Nが大きいとRFの流れが悪くなり、beamを安定に加速できなくなる。

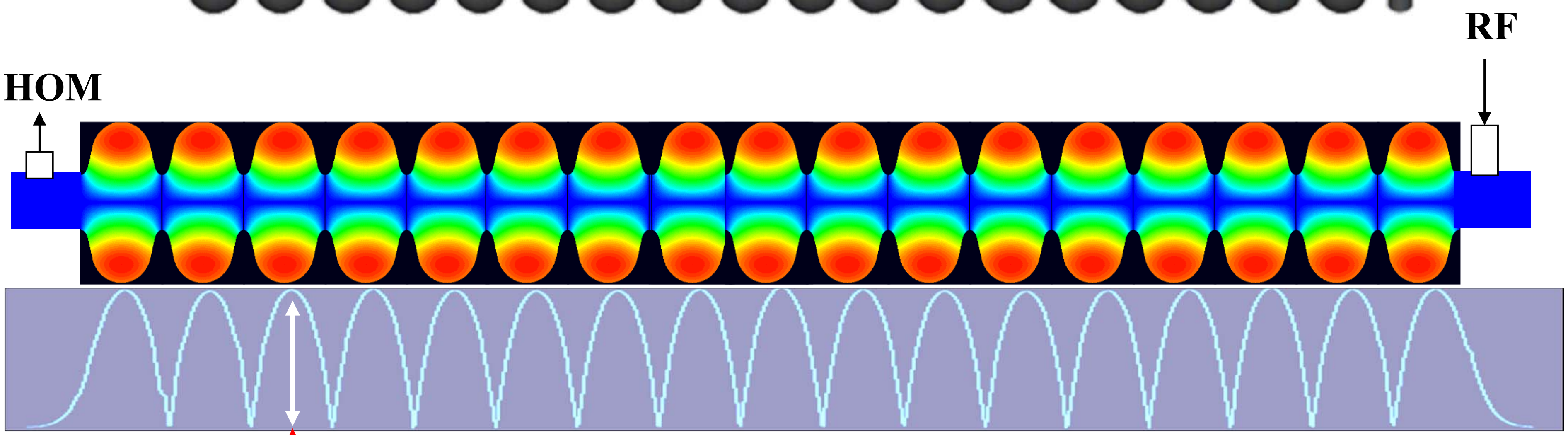
Comparison of HOM modes between Single cell and 9-cell cavity







How to decide the number of cells



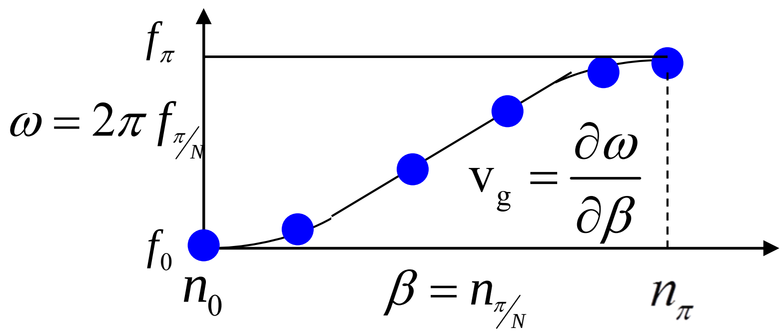
$$\frac{\Delta A_i}{A_i} = a_{ff}\,\frac{\Delta f_i}{f_i} = \frac{N^2}{k_{cc}} \cdot \frac{\Delta f_i}{f_i}$$

Beam pipe has no acceleration beam.
BP reduce the efficiency.
Multi-cell is more efficient.

N: number of cells

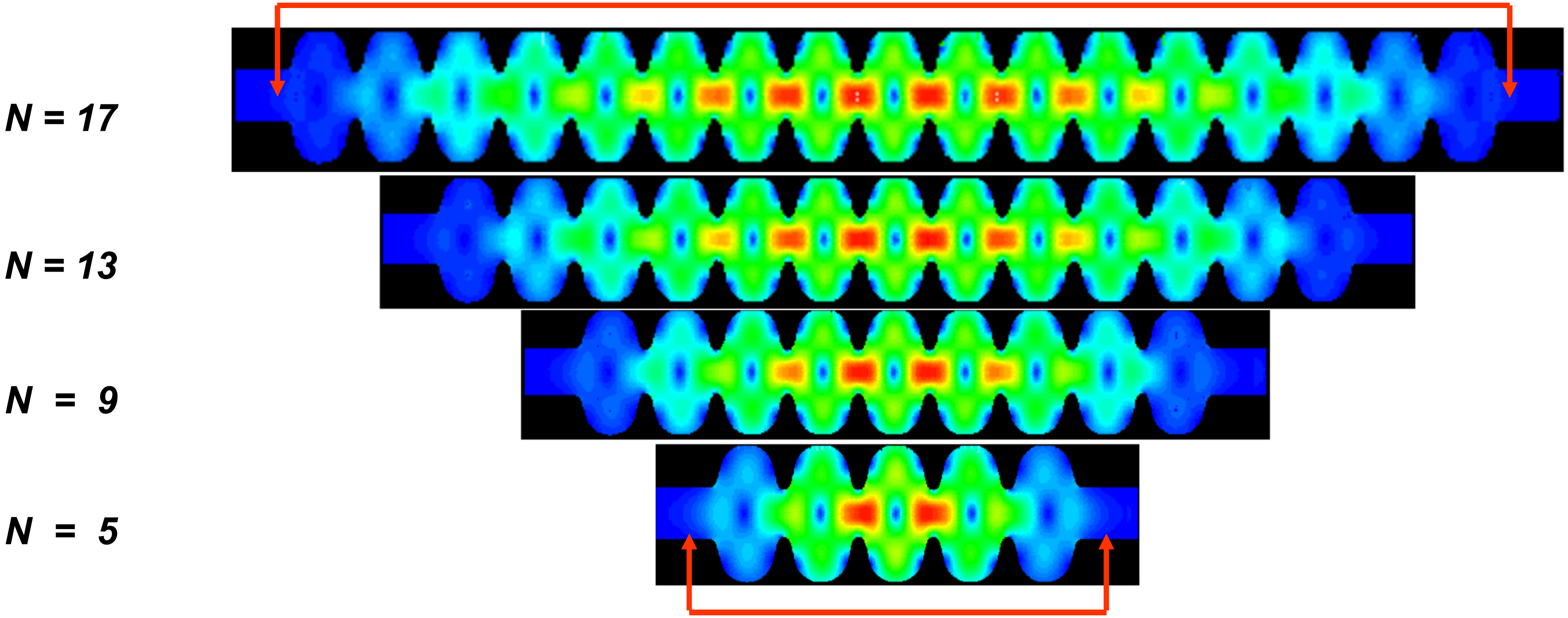
Field flatness factor : $a_{ff} = \dfrac{N^2}{k_{cc}}$

Cell to cell coupling : $\mathrm{k}_{\mathrm{cc}} = 2 \cdot \dfrac{f_\pi - f_0}{f_\pi + f_0}$



HOM trapping vs. N

No fields at HOM couplers positions, which are always placed at end beam tubes



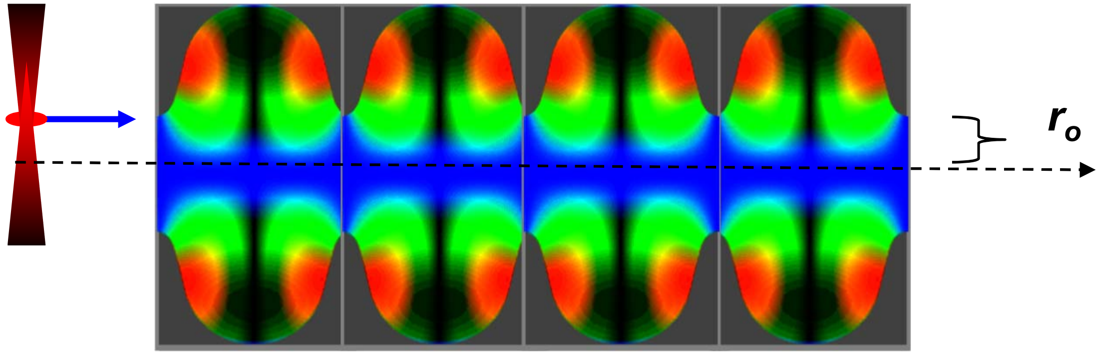
e-m fields at HOM couplers positions

Smaller number of cells is easy to take out HOMs.

4.4 HOM (Higher Order Mode)

The beam will excite HOM modes,
if it passes off beam axis of the cavity.

J.Sekutwitz's Slide



*The amount of induced energy by charge **q** is:*

$$\Delta U_q = k_{\parallel} \cdot q^2 \quad \text{for monopole modes (max. on axis)}$$

$$\Delta U_q = k_{\perp} \cdot q^2 \quad \text{for non monopole modes (off axis)}$$

where $k_{\parallel}$ and $k_{\perp}(r)$ are loss factors for the monopole and transverse modes respectively.

The induced E-H field is a superposition of <u>cavity eigenmodes</u> having the component of the <u>electric field along the trajectory</u>.

HOM Problem

J.Sekutwitz's Slide

Two kind of phenomena can limit performance of a machine due to the beam induced HOM power*:*

- *Beam Instabilities and/or dilution of emittance*
- *Additional cryogenic power and/or overheating of HOM couplers output lines*

Beam instabilities and/or dilution of emittance

Transverse modes (dipoles) causing emittance growth+ monopoles causing energy spread
This is mainly problem

in linacs*: TESLA or ILC, CEBAF, European XFEL, linacs driving FELs.*

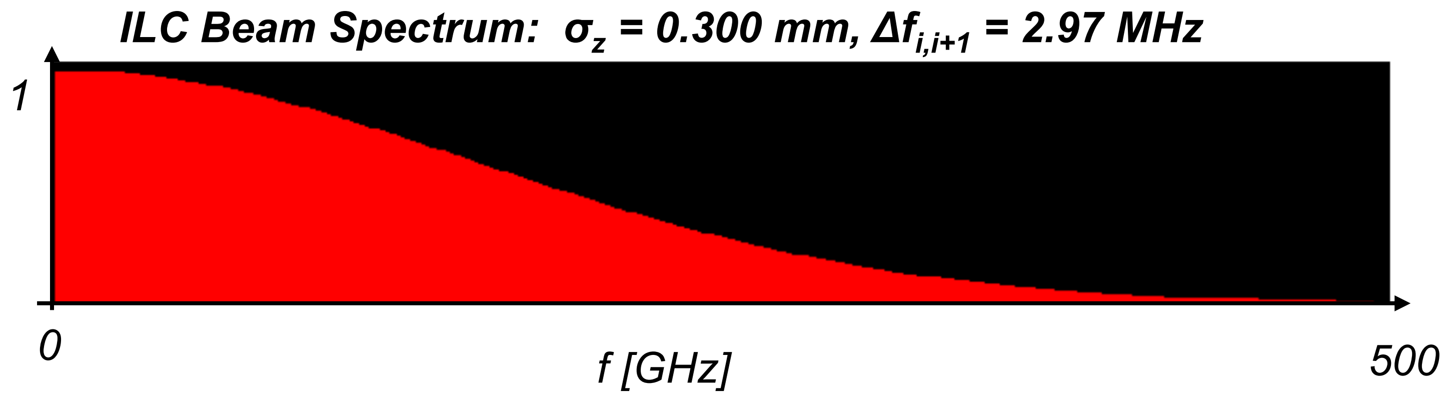
Additional cryogenic power and/or overheating of HOM couplers output lines

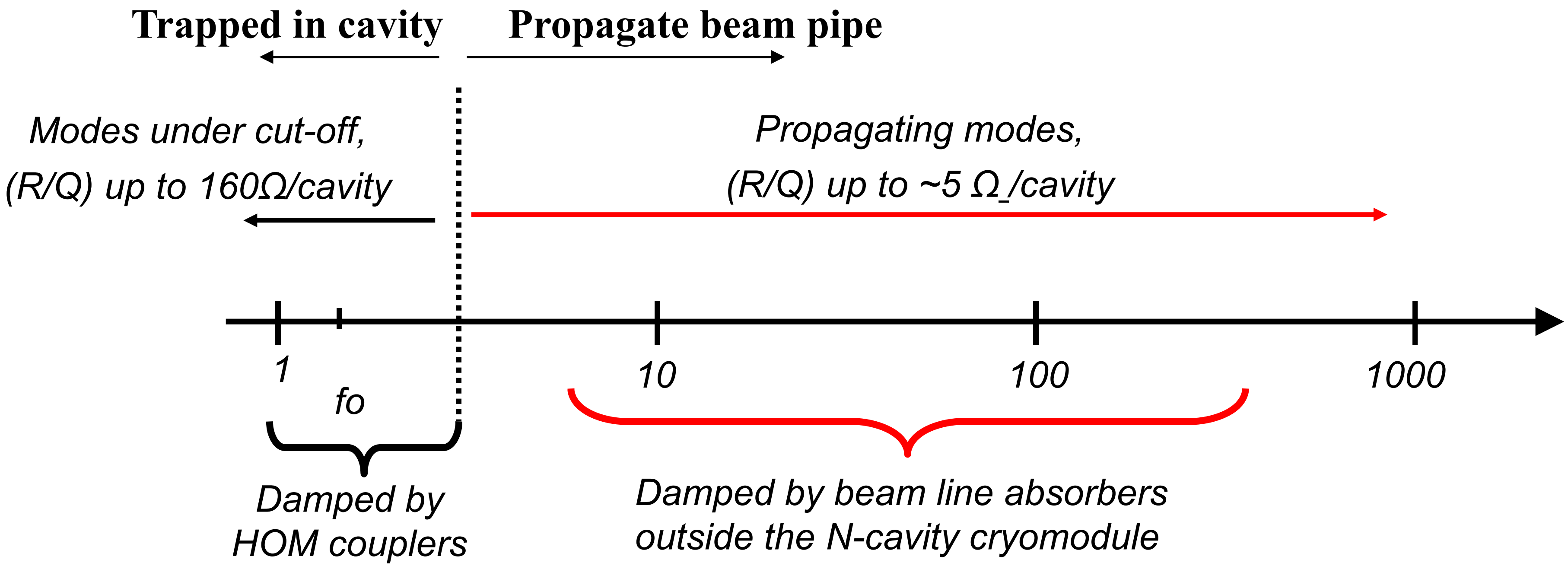
Monopoles having high impedance on axis are excited by the beam and store energy which must be coupled out of cavities, since it causes additional cryogenic load, and induces energy spread.
This is mainly problem

in high beam current *machines: B-Factories, Synchrotrons, Electron cooling.*

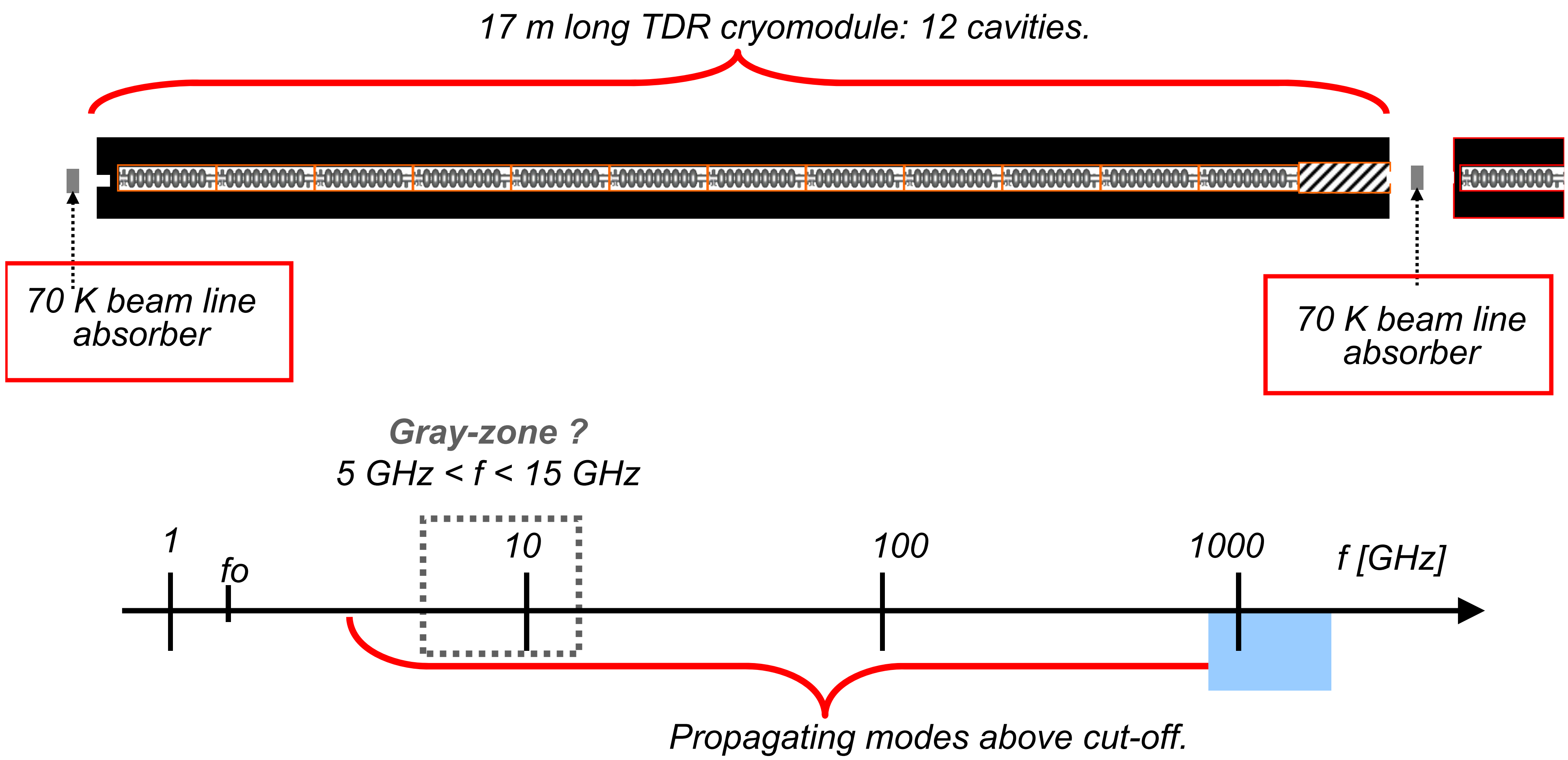
HOM modes has to be taken out from the cavity through HOM coupler.

Spectra of accelerated beams, which bunches are shorter than 1 mm, extend to hundreds of GHz.





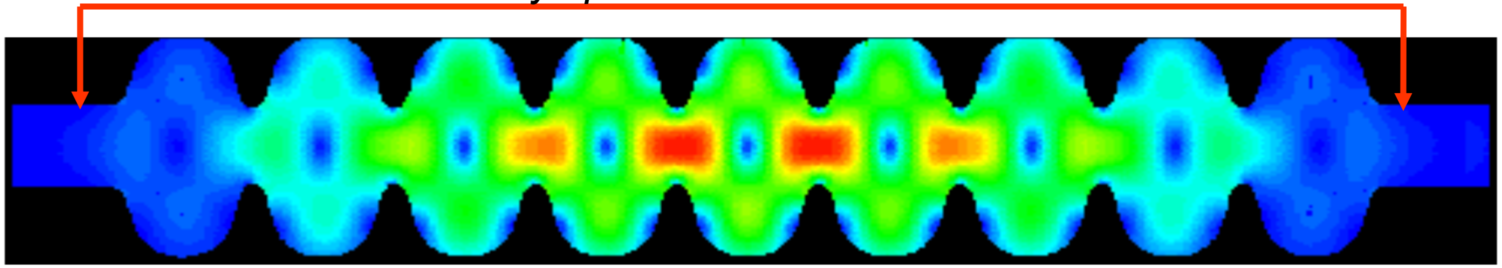
Damping the HOM with higher frequency over than Cut-off frequency



Attention to Trapped modes damping through HOM coupler

HOM couplers limit RF-performance of sc cavities when they are placed on cells

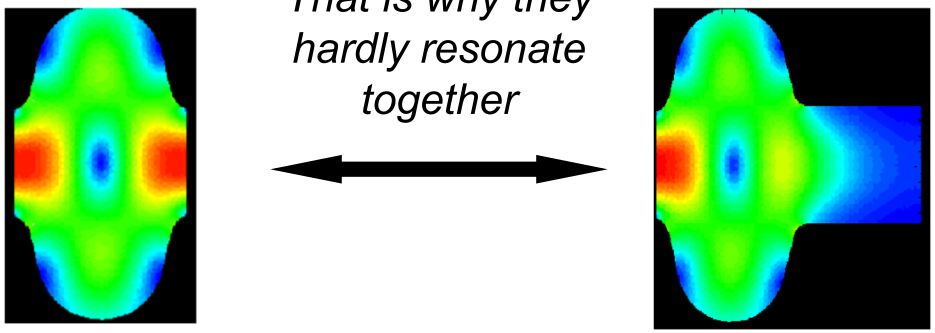
no E-H fields at HOM couplers positions, which are always placed at end beam tubes



The HOM trapping mechanism is similar to the FM field profile unflatness mechanism:

- *weak coupling HOM cell-to-cell,* $\boldsymbol{k}_{cc,HOM}$
- *difference in HOM frequency of end-cell and inner-cell*

f = 2385 MHz

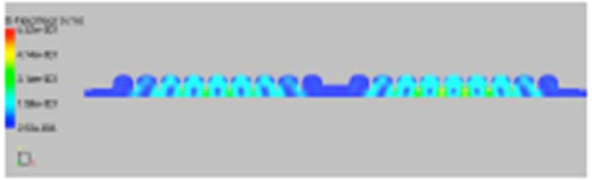


That is why they hardly resonate together

f = 2415 MHz

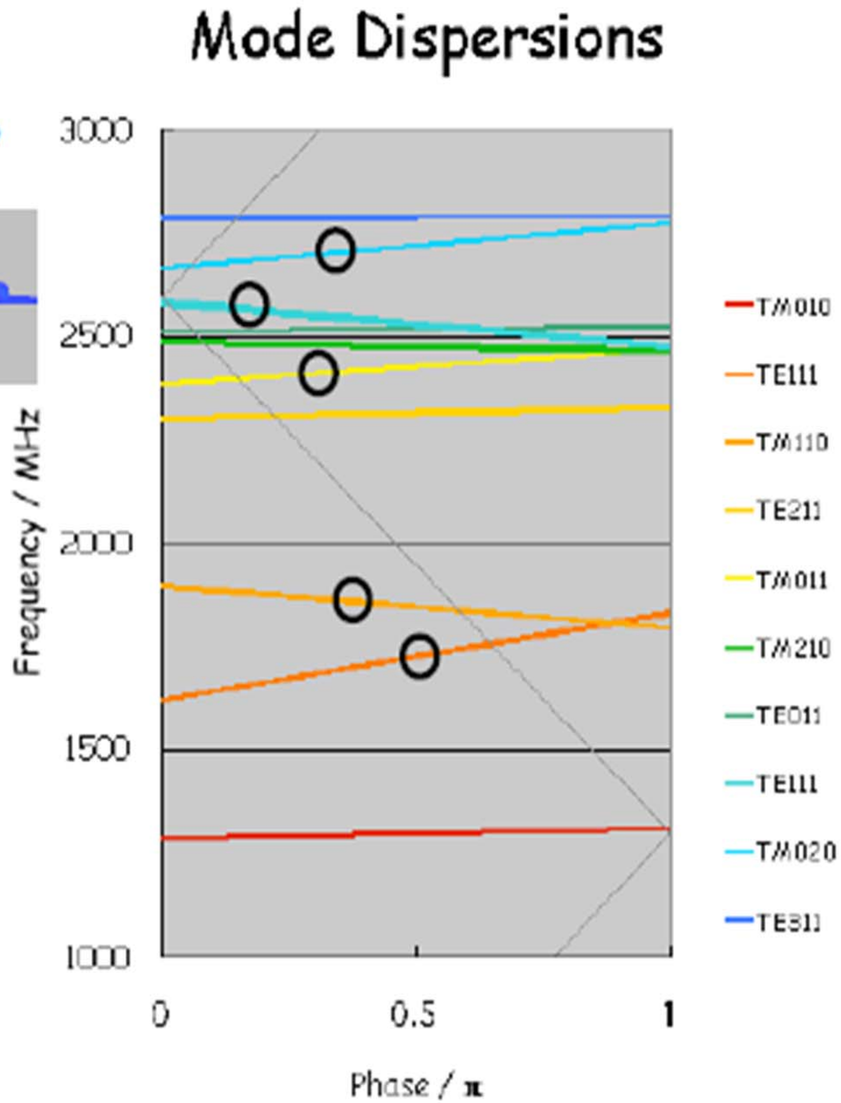
Mode Dispersions in Ichiro Cavity

Trapped Dipole Mode ?



17 V/(pCm²)

Mode crossingがある場合Trapped modeが形成される可能性が強い。



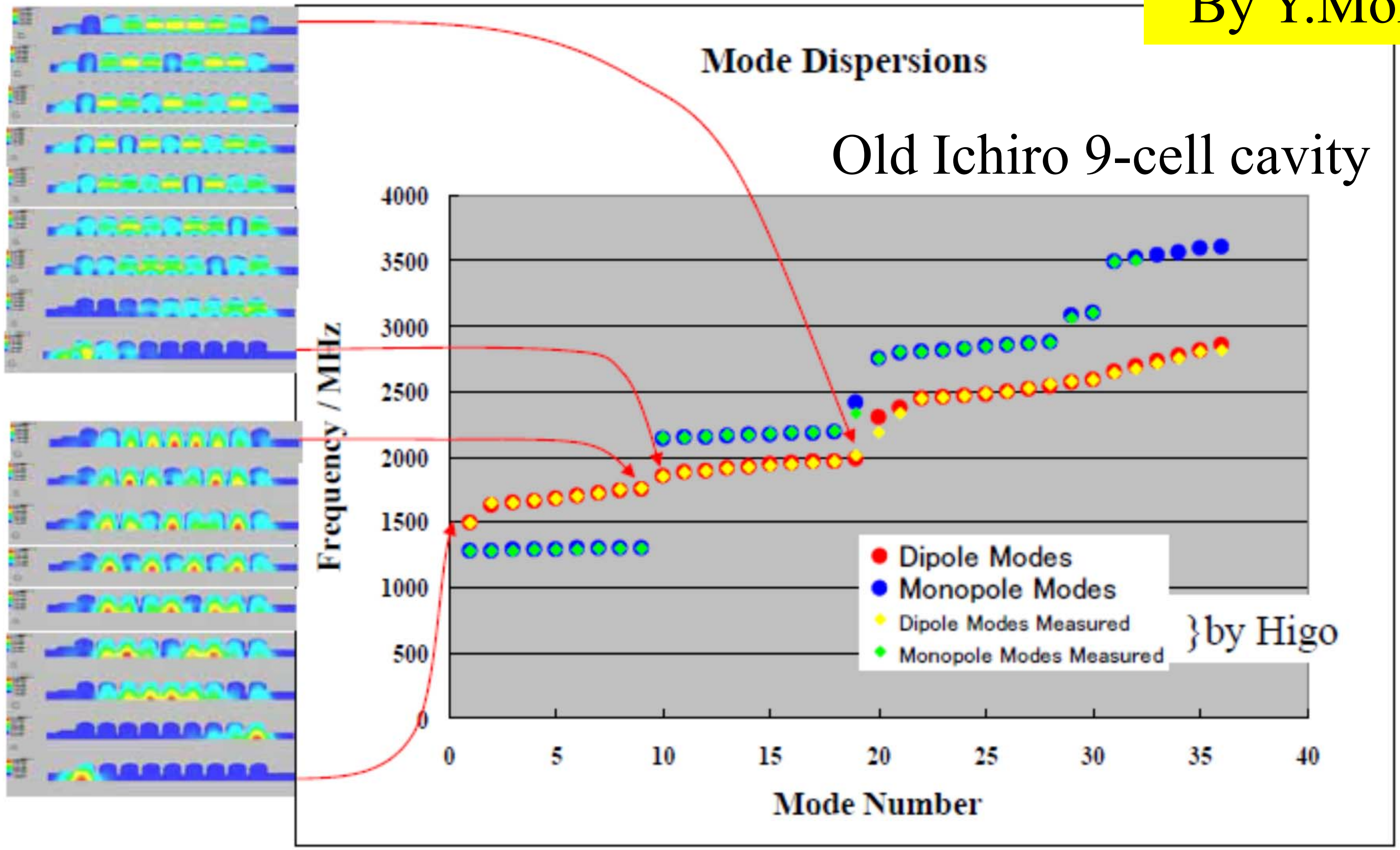
Monopole Modes
~ 3500 MHz
Dipole Modes
~ 2500 MHz

Surveyed up to now

Mode Dispersions

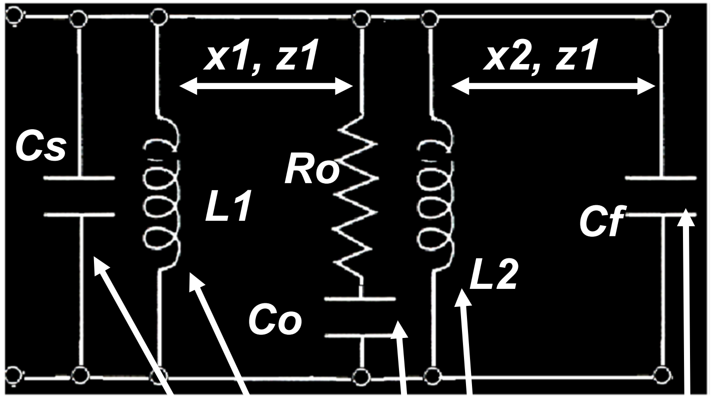
Simulated for 1st 9-Cell Structure without Coupler Ports
(Beam Ports Terminated)

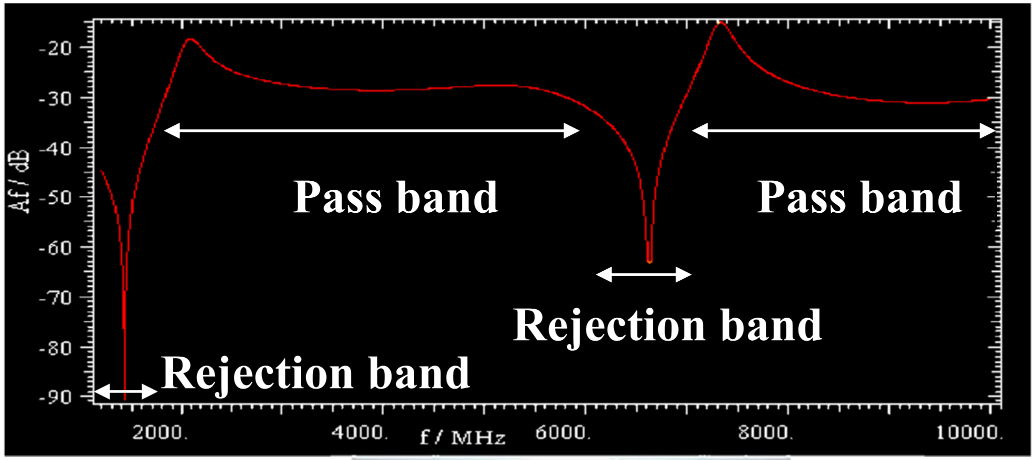
By Y.Morozumi

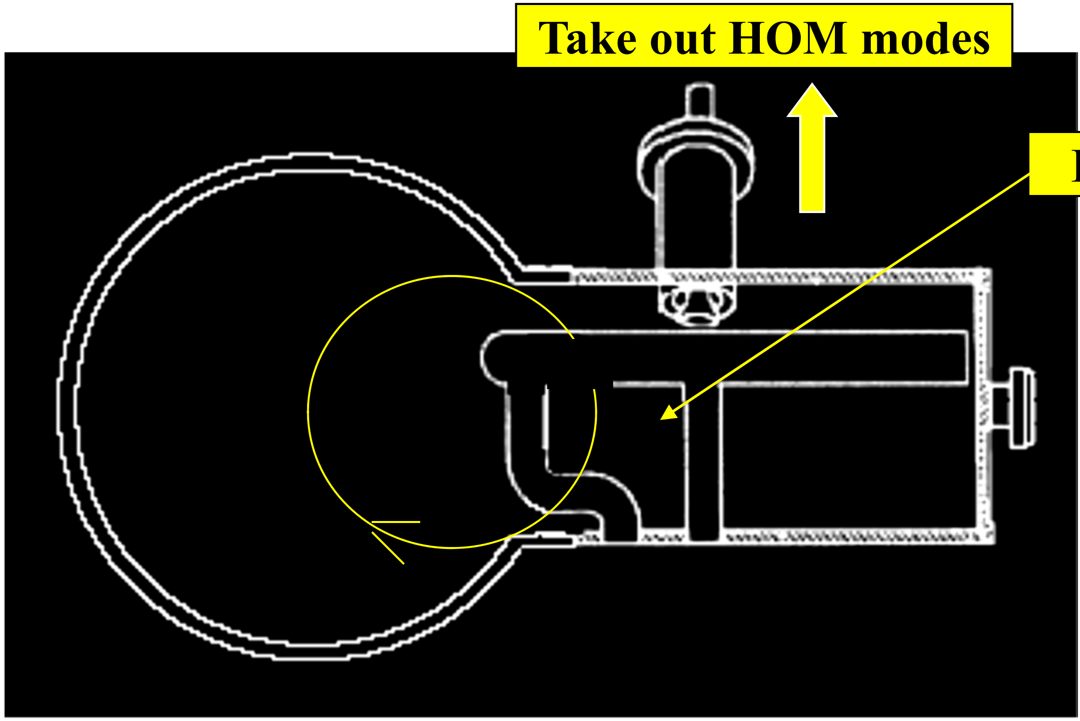


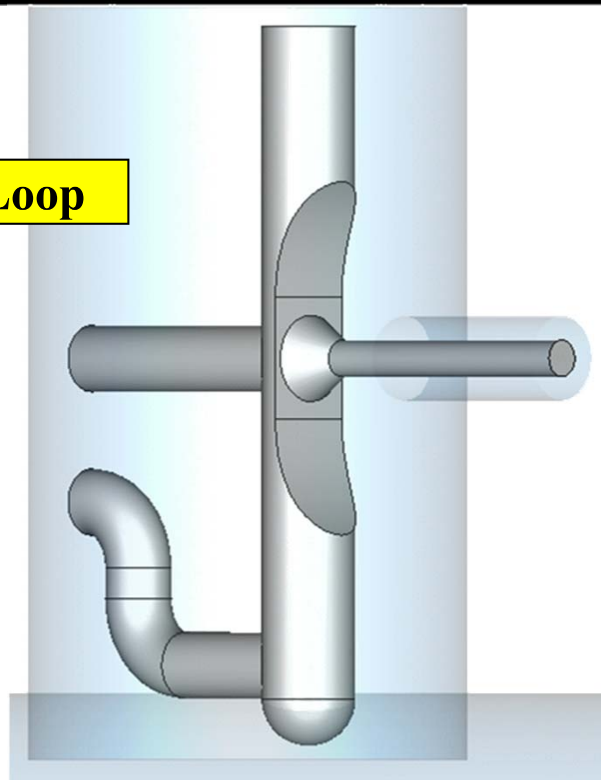
4.5 HOM Coupler

The TESLA –like HOM couplers are nowadays designed in frequency range: 0.8-3.9 GHz

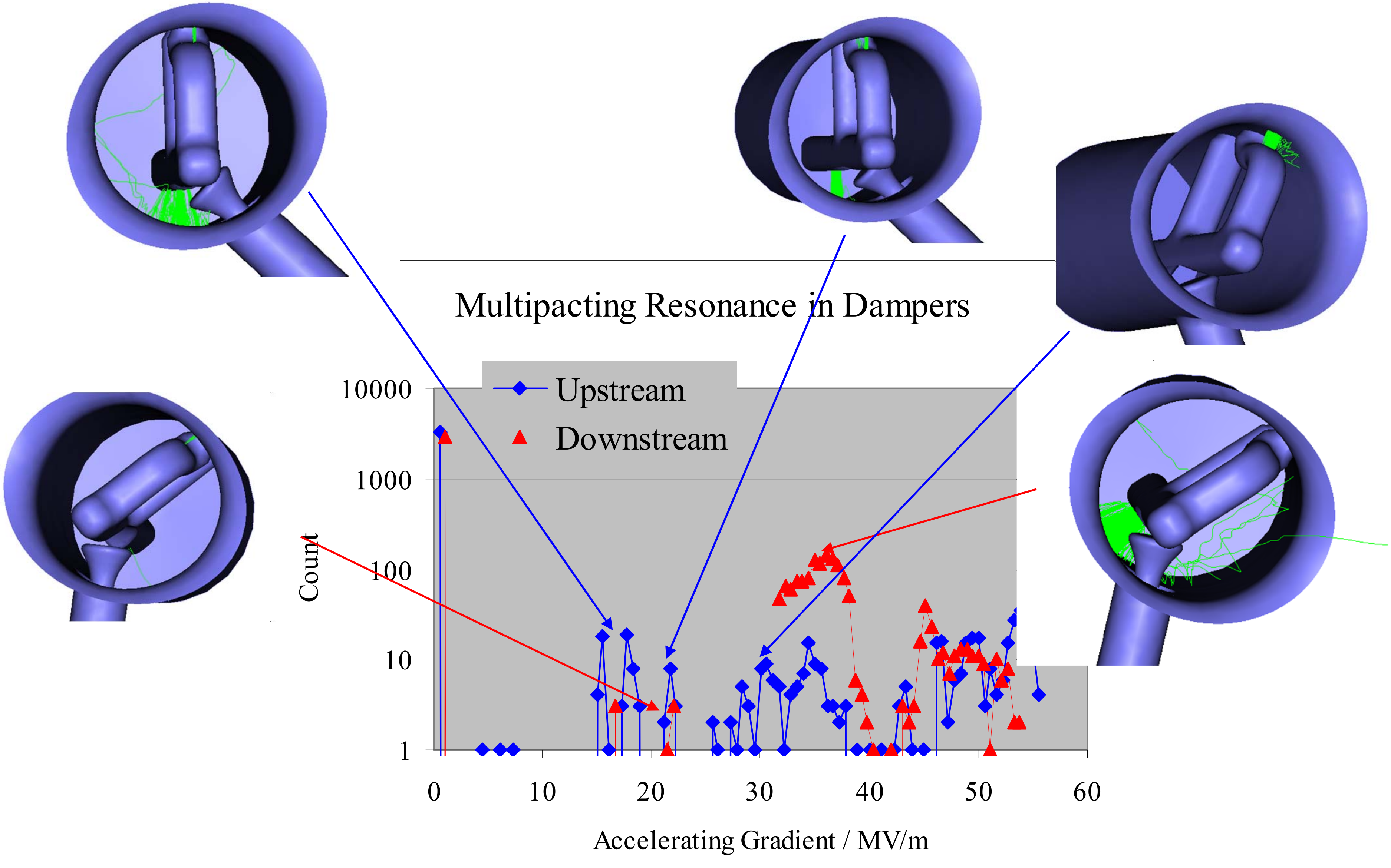






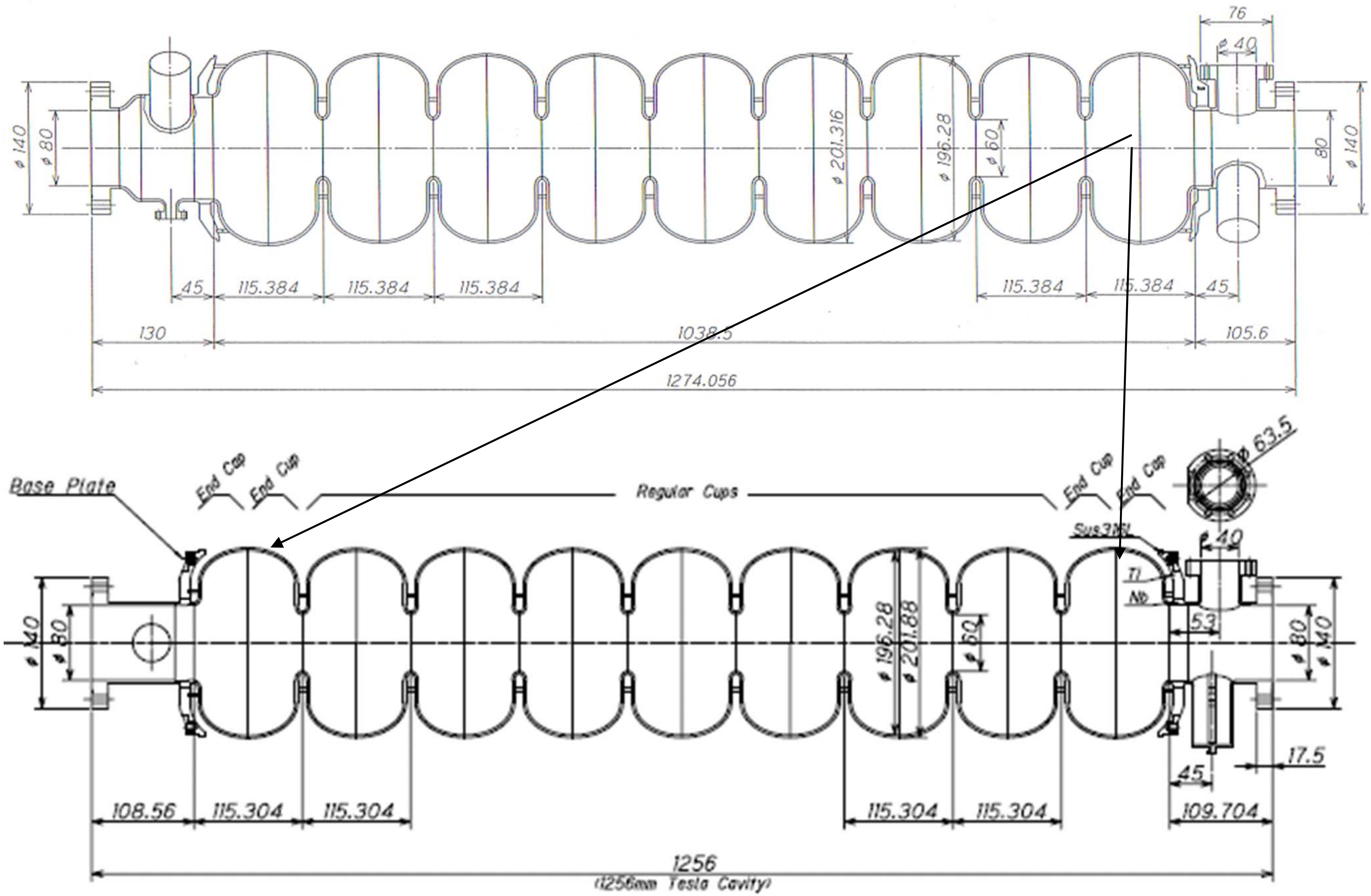


Multipacting in Current HOM Dampers

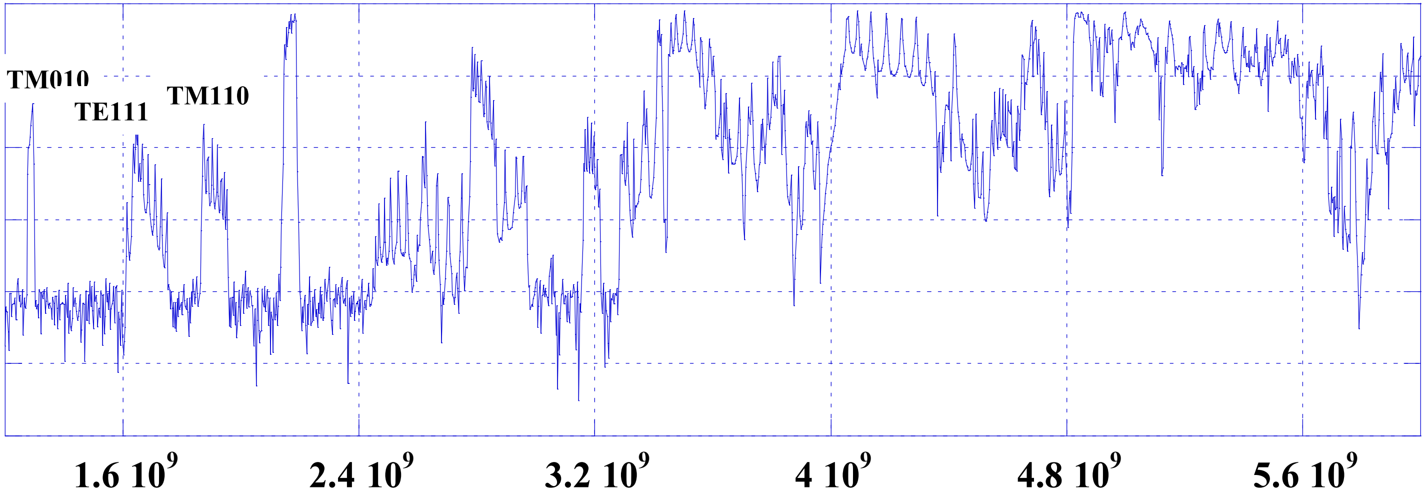


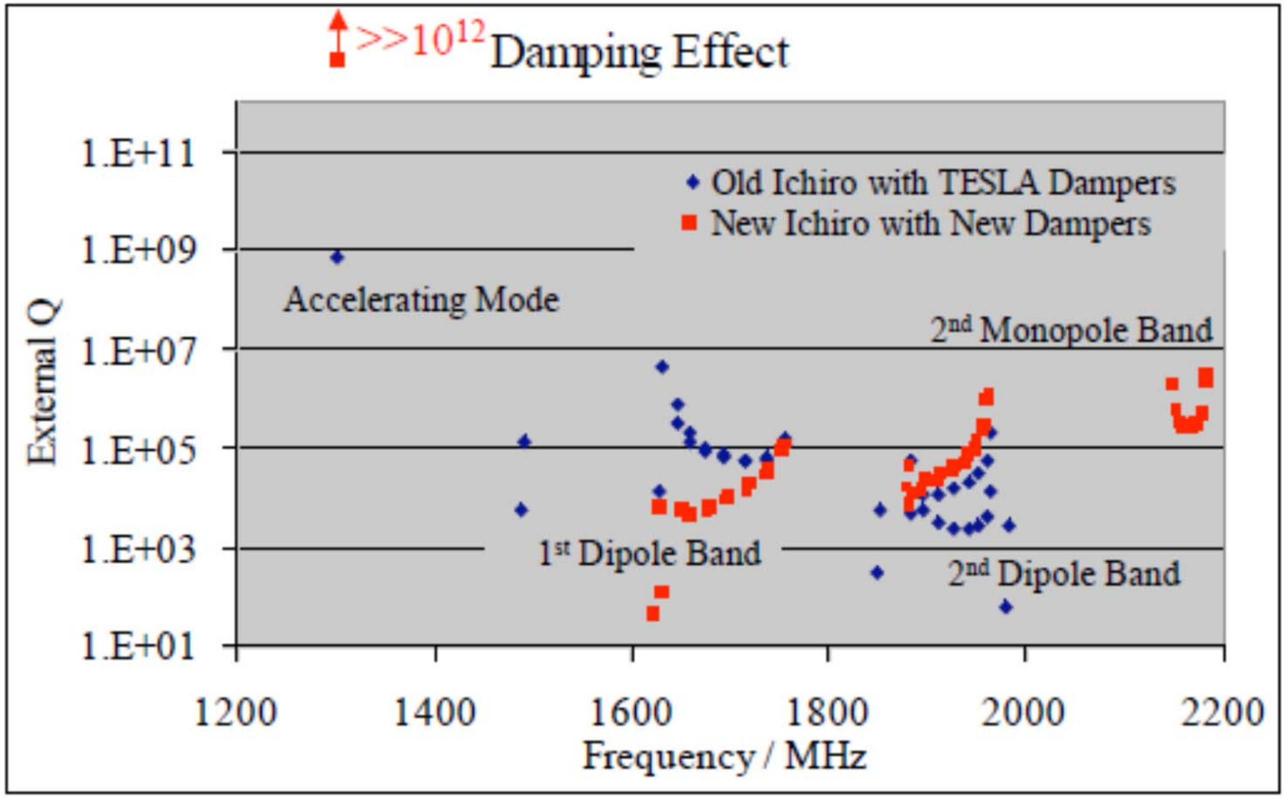
Might be processed out easily, Under redesigning for new Ichiro cavity

Re-designed Cavity Shape: Old Ichiro/New Ichiro



HOM modes in New Ichiro 9-cell Cavity





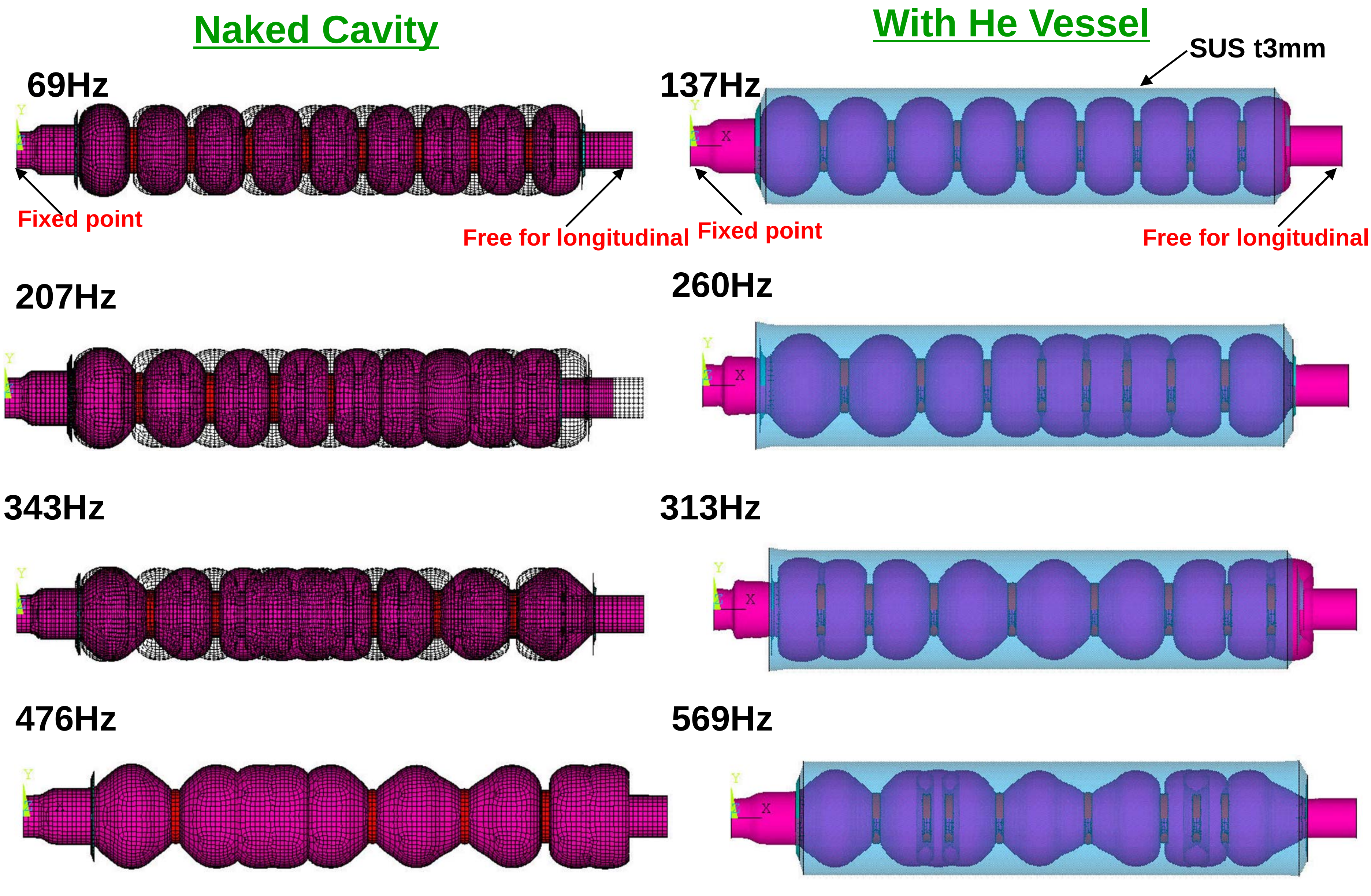
Cavity Design comparison with TESLA

	TTF	STF 45MV/m	Advantage/Disadvantage
Operation gradient	23.4MV/m 35MV/m XFEL TESLA800	45MV/m	1-1.1 TeV in 40km tunnel. If superstructure, 33km for 1TeV
Cavity shape	Iris ϕ70 Beam pipe ϕ78 cell taper 14^{O}	Iris ϕ60 Beam pipe ϕ80/ϕ108 cell taper 0^{O}	Tighter tolerance due to severe wake field
R/Q[Ω]	1036	1144	10% higher electric efficiency
Ep/Eacc	2.00	2.31	
Hp/Eacc[Oe/(MV/m)]	42.6	37.8	Eacc max 46-49MV/m
Cell-to-cell coupling	1.86	1.55	F-flatness more sensitive on fabrication error
Tolerance[μm]	250	170	
Sensitivity for Lorenz detuning[Hz/(MV/m)^2]	1	1.18	Wide detuning range
Lorenz detuning[Hz]	200 600	~1500	Need wide range tuner
He jacket material	5t Ti	3t SUS316L	No matter with Japanese high pressure code
RF transmission power[kW]	240 350	500	Need higher input coupler

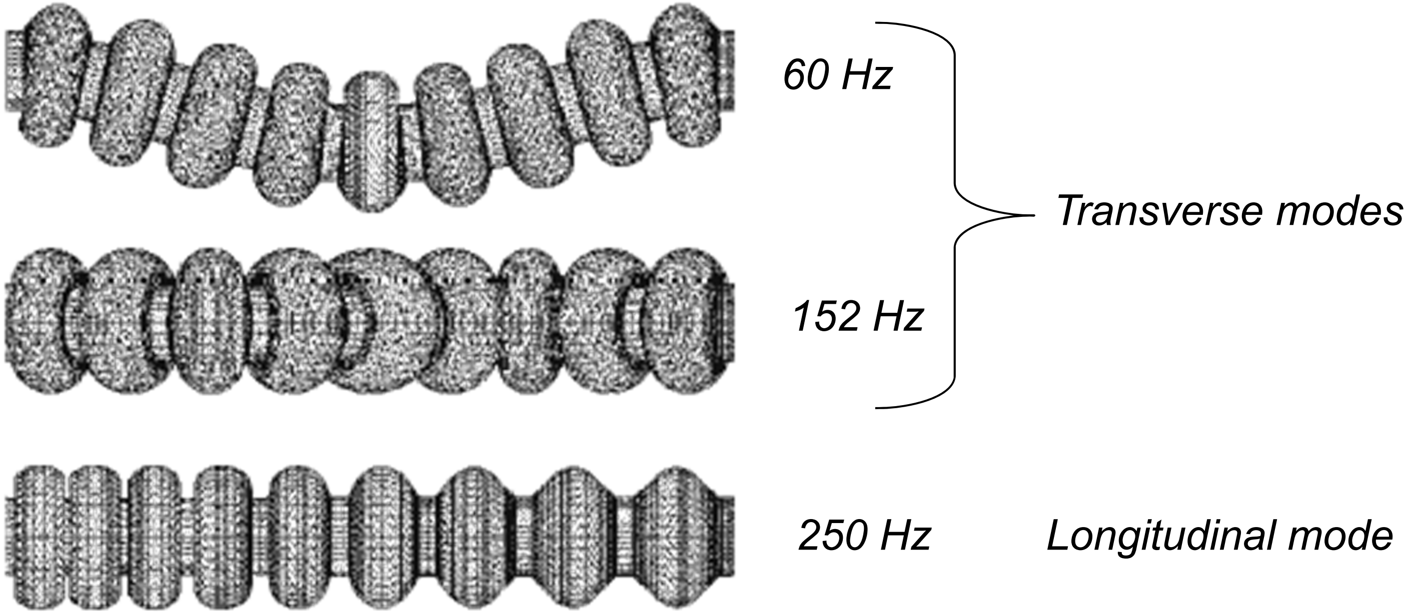
4.6 Mechanical Analysis

Calculation of longitudinal mechanical resonance w/wo He vessel

by H.Yamaoka



Mechanical Resonance of a multi-cell cavity



TESLA structure

The mechanical resonances modulate frequency of the accelerating mode.
Sources of their excitation: vacuum pumps, ground vibrations…

Mechanical Resonance Excitation

$$m\frac{d^2x}{dt^2}+\lambda\frac{dx}{dt}+kx=mf\sin(\omega t):\quad \text{Forced oscillation}(\omega)$$

$$x=ae^{-\mu t}\sin(\omega_1 t+\phi_1)+A\sin(\omega t-\varphi)$$

$$\omega_0=\sqrt{\frac{k}{m}},\ \omega_1=\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}},\ \mu=\frac{\lambda}{2m},$$

$$A=\frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}},\quad \tan(\varphi)=\frac{2\mu\omega}{\omega_0^2-\omega^2}$$

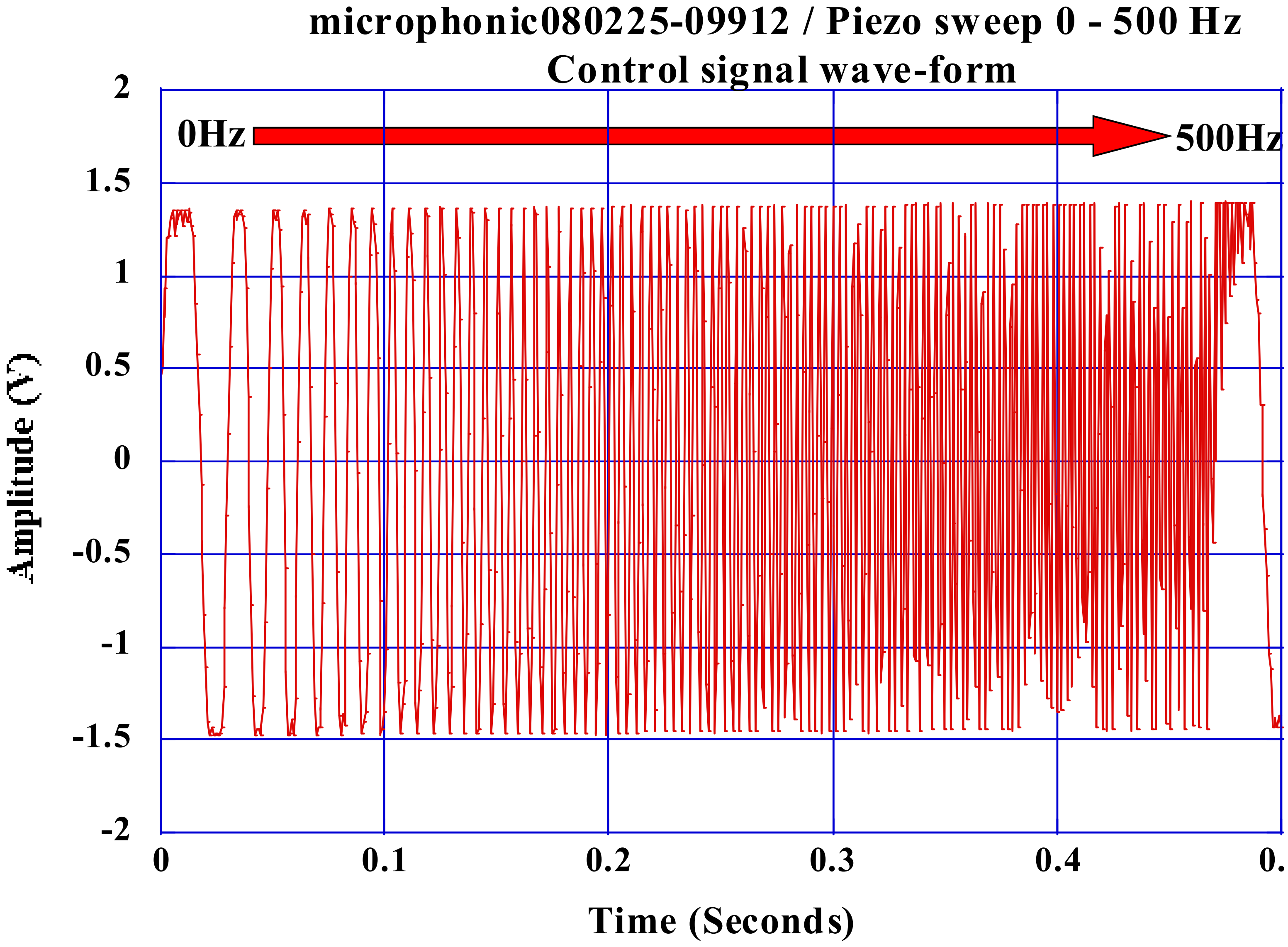
振幅は $\omega=\omega_r=\sqrt{\omega_0^2-2\mu^2}\approx\omega_0$ で最大になる。

$\mu^2\ll\omega_0^2,\quad \omega_0\cong\omega_1\cong\omega_r$ を使ってAをもう少し書き直すと

$$\text{if}\,\omega\sim\omega_r,\ \boxed{A^2(\omega)\approx\frac{A_r^2\cdot\mu^2}{(\omega-\omega_r)^2+\mu^2}},\ A_r=\frac{f}{2\mu\omega_1},\ 2\mu\approx\omega_+-\omega_-=\Delta,\ \boxed{Q_m\equiv\frac{\omega_r}{\Delta}}$$

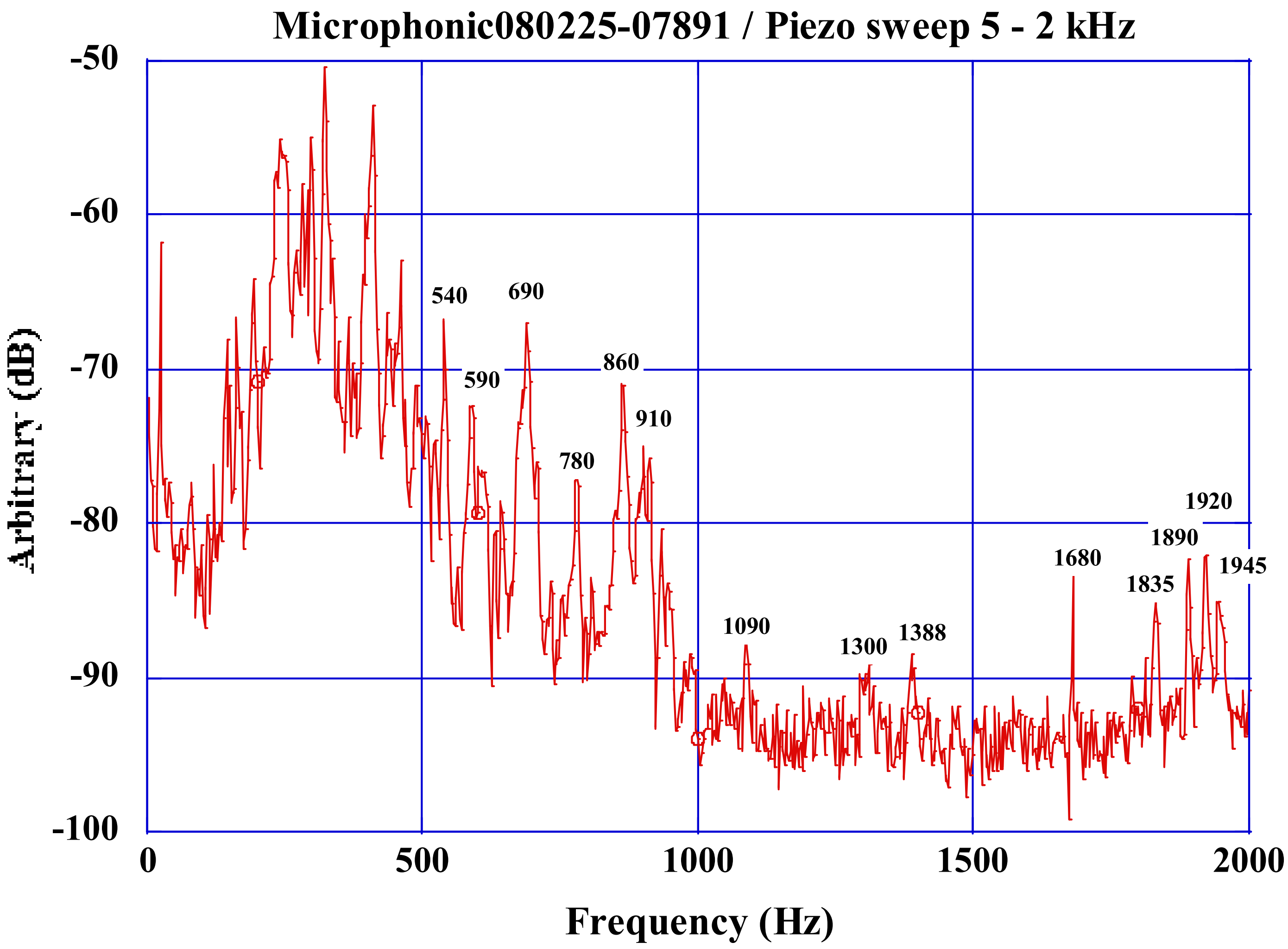
Piezo Excitation by f-sweep mode (Example)

Example of Piezo excitation



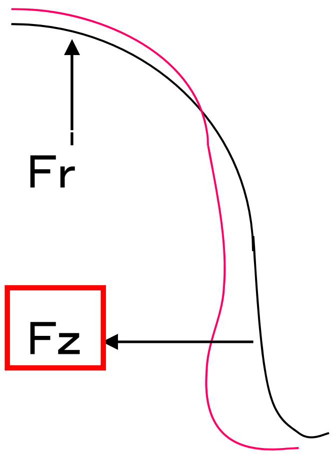
Mechanical Resonance Scan (0 – 2kHz)

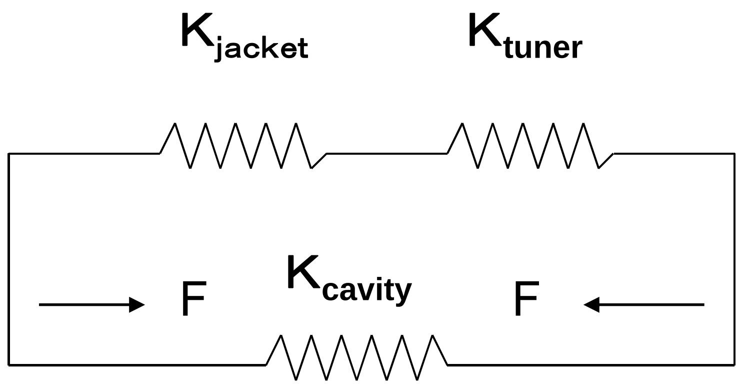
FFT signal



Tow components of Lorentz Deformation

Noguchi's slide in the 1st ILC school





$$\Delta f = \sum_{\text{mode}} a_k \, \delta f_k \approx \sum_{\Delta l = 0} a_k \, \delta f_k + \frac{d\,f}{d\,l}\frac{d\,l}{d\,F} F$$

$$= A\,E_{acc}^2 + \boxed{\frac{d\,f}{d\,l}\frac{B\,E_{acc}^2}{K_S}}$$

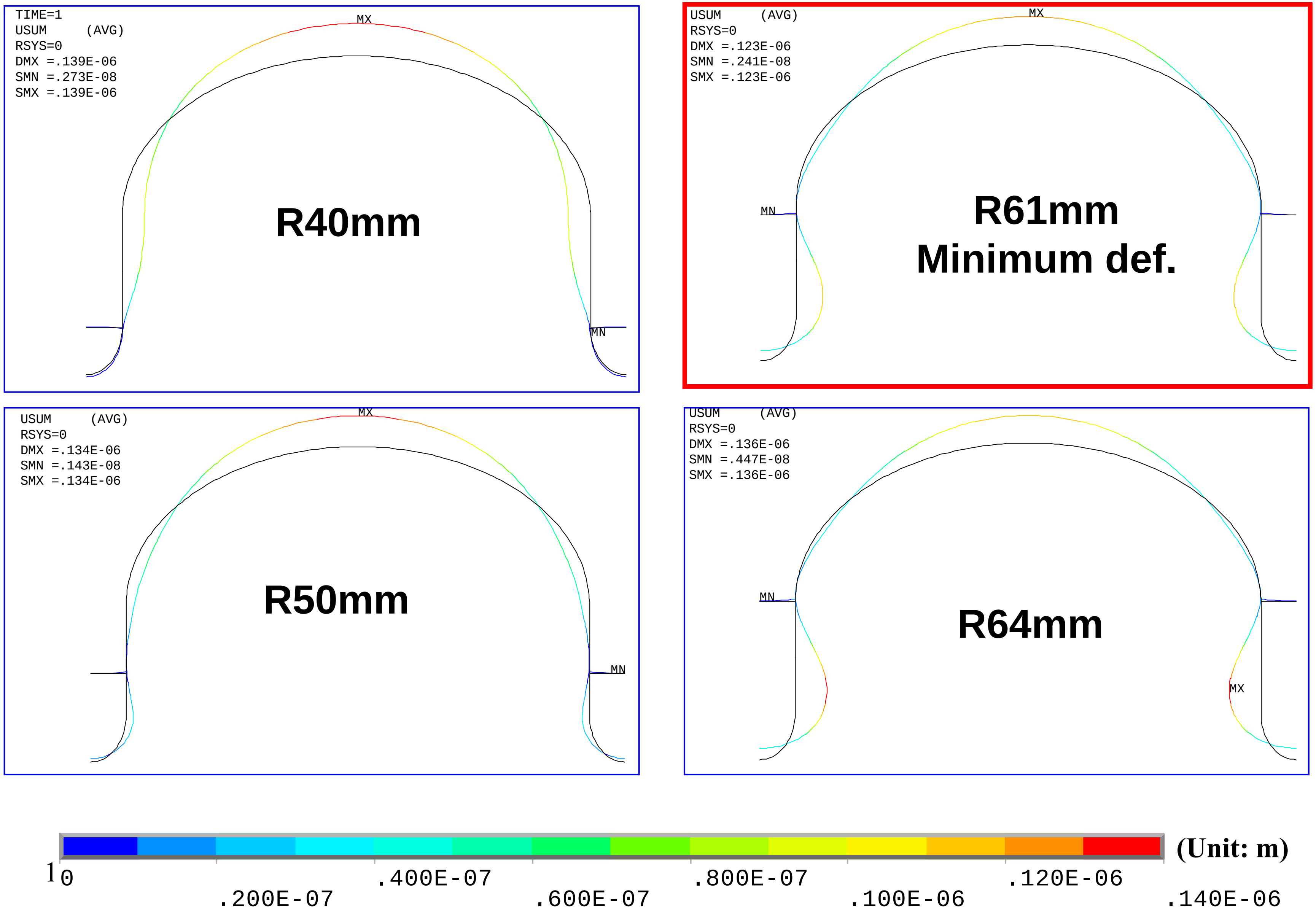
Rigid Stiffness at Jacket and Tuner are also Very important against the Lorentz Detuning.

		TESLA Blade	STF Slide Jack	STF Ball Screw
A	Hz/(MeV/m)^2	0.5	0.5	(1.2)
B	N/(MeV/m)^2	0.047	0.047	0.051
df/dl	Hz/ μ m	320	320	370
dF/dl	N/ μ m	3	3	1.8
KS	N/ μ m	13	80	60
Kjacket	N/ μ m	26	96	58
Ktuner	N/ μ m	26	500	1700
Δf (30MV/m)	Hz	1490	620	(1360)
Fine Tuning Stroke	μ m	3.7	1	2.9

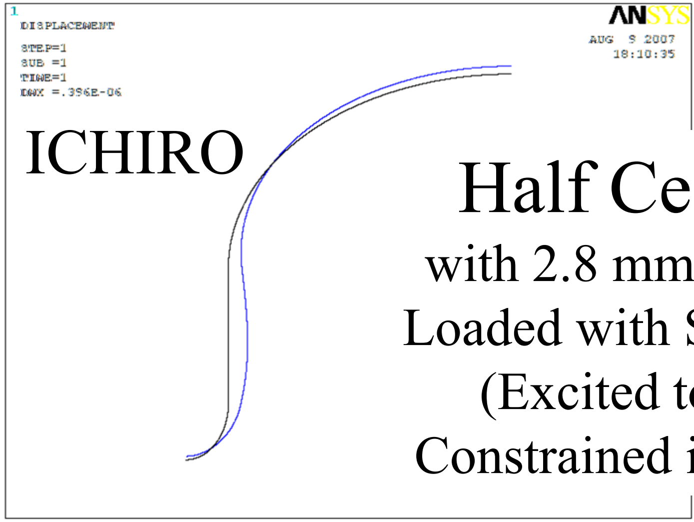
Optimization of Stiffener Location against Lorentz Detuning

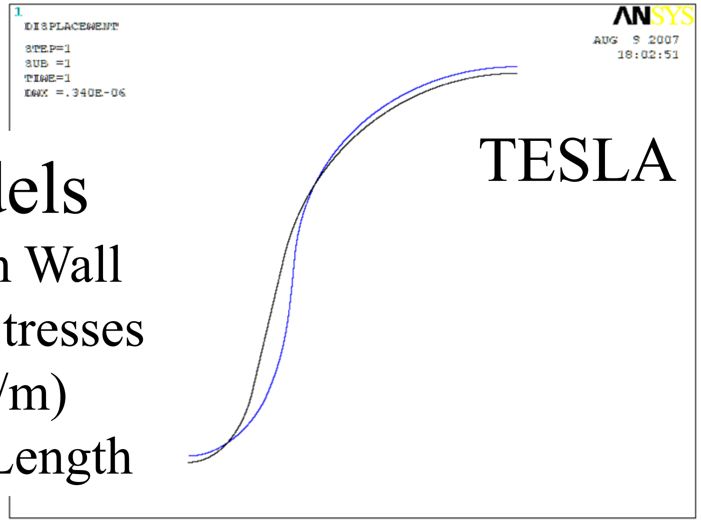
Eacc=38MV/m

by H.Yamaoka



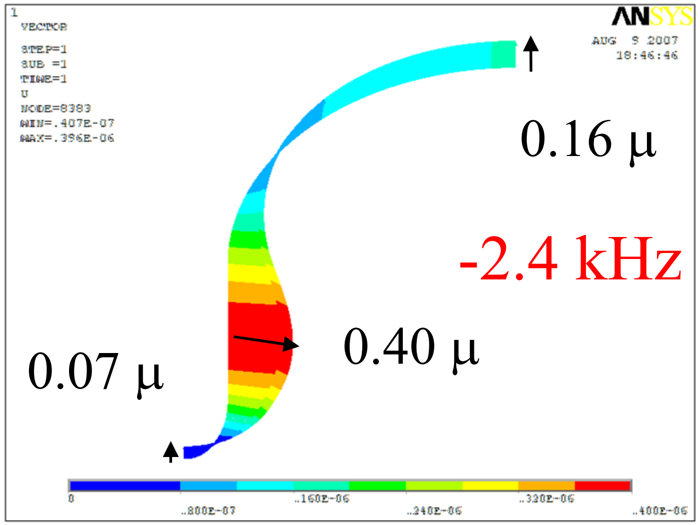
Comparison of Lorentz Force Deformation between different cell shapes

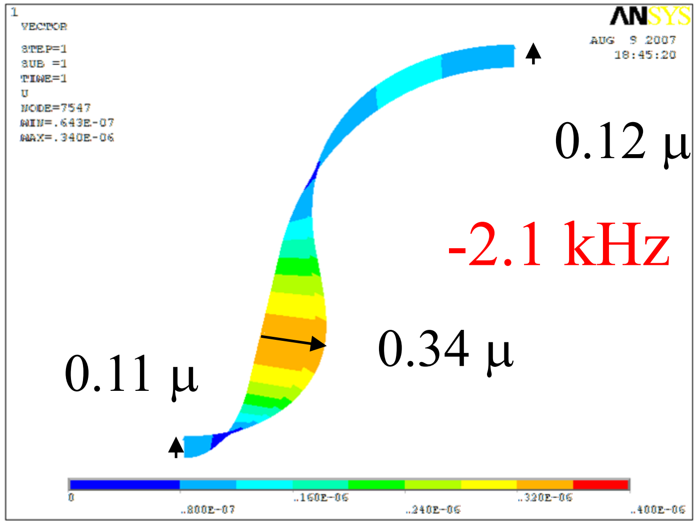




Half Cell Models

with 2.8 mm Niobium Wall
Loaded with Surface Stresses
(Excited to 40 MV/m)
Constrained in Axial Length





4. Summary

- ここでは、空洞設計のエッセンスを理解してもらうために空洞設計法を与えた。
- 超伝導空洞では、**Q**値が高いので定在波を使う。
- 超伝導空洞の設計と言えども常伝導の場合と全く同じである。**Maxwell** 方程式を与えれた境界条件の下で固有値問題を解くことである。
- 実際の空洞形状では解析的に処理できないので、有限要素法による電磁場計算コードを使う。**SUPERFISH, MAFIA**等が一般に使われる。